A FOUR-PERSON CHESS-LIKE GAME WITHOUT NASH EQUILIBRIA IN PURE STATIONARY STRATEGIES

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In this paper we give an example of a finite positional game with perfect information and without moves of chance (a chess-like game) that has no Nash equilibria in pure stationary strategies. In this example the number n of players is 4, the number p of terminals is 5; furthermore, there is only one directed cycle.

On the other hand, it is known that a chess-like game has a Nash equilibrium (NE) in pure stationary strategies if: (A) $n \le 2$; or (B) $p \le 3$ and (C) any infinite play is worse than each terminal for every player; or (D) each of n players controls a unique position; or (E) there exist no directed cycles.

It remains open whether a NE-free chess-like game (with at least one directed cycle) may exist in each of the following four cases: (A') $n = 3$; (B') $2 \le p \le 4$; (C') $n > 2$, $p > 3$, and condition (C) holds; (D') each of n players controls at most 2 positions.

In our example $n = 4$, $p = 5$, condition (C) does not hold, and there is a player controlling 3 positions.

Key words: positional game, stochastic game, chess-like game, perfect information, move of chance, Nash equilibrium, terminal position, directed cycle.

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1. Introduction

Zermelo gave his seminal talk on solvablity of chess in pure strategies [40] as early as in 1912. Later, König [29] and Kalman [28] strengthen this result showing that in any two-person zero-sum chess-like game there exist pure stationary uniformly optimal strategies producing a subgame perfect saddle point.

Let us note that the same position can appear several times in the game of chess, or in other words, the corresponding (directed) graph has (directed) cycles.

A *chess-like game* is a finite n-person positional game with perfect information and without moves of chance. The set of its $q = p + 1$ outcomes $A = \{a_1, ..., a_p, c\}$ consists of p terminals, from which there is no move; furthermore all infinite plays are assumed to be equivalent and to form a unique special outcome c.

The following assumption will play an important role:

✦ (C): Any infinite play is worse than any terminal for each player.

In 1950, Nash introduced his fundamental concept of equilibrium for n-person games [35, 36]. After this, it

became natural to ask, whether one can extend the above solvability results to the n-person chess-like games, replacing the concept of a saddle point by the more general concept of Nash equilibrium (NE), assuming (C) or not; or in other words,

✦ (Q): Whether a chess-like game has a NE in pure stationary strategies?

✦ (Q_C): Whether a chess-like game satisfying (C) has a NE in pure stationary strategies?

Obviously, the positive answer to (Q) would imply the positive answer to (Q_C). Yet, in this paper, we answer (Q) in the negative, while (Q_C) remains still open.

> ***Remark 1.*** *Both questions* (Q) *and* (Q_C) *appear in Table 1 of* [3]. *This table contains 16 pairs of questions exactly one of which requires* (C) *in each pair; 30 questions (15 pairs) are answered in [3]. Among the 30 answers, 22 are negative and 8 are positive, but interestingly, no answer ever depends on* (C), *that is, in each of these 15 pairs either both answers are negative, or both are positive. Our questions* (Q) *and* (Q_C) *form the 16th pair;* (Q) *will be answered in the negative in this paper, while* (Q_C) *remains open. Yet, taking the above observations into account, we second* [26] *and conjecture that the answer to* (Q_C) *is the same as to* (Q), *that is, negative. However, the example for* (Q_C) *might be larger than our example for* (Q) *and more difficult to find out.*

In Section 3 we will construct a NE-free chess-like game with four players, $n = 4$, and five terminals, $p = 5$, or in other words, six outcomes, $q = 6$. Furthermore, the corresponding directed graph has a unique directed cycle.

In Section 3 of [7] it was shown that a chess-like game has a NE when $n \le 2$; see also Section 12 of [9] and Section 5 below. This result is based on an old criterion of Nash-solvability for the two-person game forms [20]. It Section 4 of [7] it was also shown that a chess-like game has a NE whenever condition (C) holds and $p \le 2$. In [10] this result was strengthened: the bound $p \le 2$ was replaced by $p \le 3$. It remains still open whether a NE-free chess-like game exists (i) for $n = 3$, or (ii) for $2 \le p \le 4$, or (iii) for some n and p provided (C) holds.

> ***Remark 2.*** *It was shown in* [15] *that for each* $\epsilon > 0$, *a subgame perfect* ϵ-NE *in pure but* history dependent *strategies exists, even for the n-person* backgammon-like games, *in which positions of chance are allowed. Moreover, it was shown in* [14, 15] *that for the* chess-like *games, the above result holds even for* $\epsilon = 0$, *that is, a standard NE exists too.*

Yet, our example shows that pure stationary strategies may be insufficient to ensure the existence of a NE in any n-person chess-like game, when $n \ge 4$; *not to mention the existence of a subgame perfect NE. For the latter case counterexamples, satisfying* (C) *with* $n = 2$ *and* $n = 3$, *were obtained earlier* [7, 1, 3].

2. Main definitions

The *backgammon-like* and *chess-like* games are finite positional n-person games with perfect information, which can and, respectively, cannot have random moves.

More precisely, such a game is modeled by a finite directed graph (digraph) $G = (V, E)$, whose vertices are partitioned into $n + 2$ subsets: $V = V_1 \cup ... \cup V_n \cup V_T \cup V_R$. A vertex $v \in V_i$ is interpreted as a position controlled by the player $i \in I = \{1, ..., n\}$, while $v \in V_R$ is a *position* of chance, with a given probabilistic distribution on the outgoing edges. Furthermore, a directed edge (v, v') is interpreted as a *move* from the position v to v'. Then, $v \in V_T = \{a_1, ..., a_p\}$ is a *terminal*, from which there is no move. We also fix an initial position $v_0 \in V \setminus V_T$.

A game is called *chess-like* if it has no positions of chance, $V_R = \varnothing$.

The digraph G may have *directed cycles* (dicycles). Recall that a position may appear several times in a backgammon or chess play. We assume that all dicycles of G form a unique outcome c of the game. Thus, the set of outcomes is $A = \{a_1, ...,a_p; c\}$.

> ***Remark 3.*** *In* [9] *a different approach was suggested* (*for* $n = 2$): *each dicycle was treated as a separate outcome. Anyway, our main example contains only one dicycle.*

To each player $i \in I$ and outcome $a \in A$ we assign a payoff (called in the literature also a reward, utility, or profit) $u(i, a)$ of the player $i \in I$ in case the outcome $a \in A$ is realized. The corresponding mapping u: $I \times A \to \mathbb{R}$ is called the payoff (reward, utility, or profit) function.

Since our main result is negative and related to chess-like games, we could restrict ourselves and the players to their strict preferences, instead of the real-valued payoffs. The *preference* of a player $i \in I$ is a complete order o_i over A. The notation $a \succ_{o_i} a'$ and $a \succcurlyeq_{o_i} a'$ mean that i prefers a to a' strictly and, respectively, not strictly. Note that the latter takes place if and only if $a = a'$. Furthermore, let $o = (o_1, ..., o_n)$ denote a *preference profile*.

A *backgammon-like game in the positional form* is the quadruple (G, D, o, v_0), where $G = (V, E)$ is a digraph,

$D: V = V_1 \cup ... \cup V_n \cup V_T \cup V_R$ is a partition of the positions, $o = (o_1, ..., o_n)$ is a *preference profile*, and v_0 is a fixed initial position. The triplet (G, D, v_0) is called a *positional game form*.

To define the *normal form* (of a chess-like game) let us introduce the concept of strategies. A (*pure and stationary*) *strategy* of a player $i \in I$ is a mapping that assigns a move (v, v') to each position $v \in V_i$. (In this paper we restrict ourselves and the players to their pure and stationary strategies, so mixed and history dependent strategies will not be mentioned or even introduced.)

A set of n strategies $s = \{s^i, i \in I\}$ is called a *strategy profile* or a *situation*. Each situation uniquely defines a *play* $P(s)$ that begins in v_0 and either ends in a terminal $a \in V_T$ or cycles. In the latter case $P(s)$ looks like a «lasso»: it consists of an initial part and a dicycle repeated infinitely. This is so, because each (pure stationary) strategy assigns the same move whenever a position is repeated and, hence, each situation s uniquely defines a move (v, v') in each non-terminal position $v \in V \setminus V_T$. Thus, we obtain a *game form*, that is, a mapping $g: S \to A$, where $S = S_1 \times ... \times S_n$ is the direct product of the sets $S_i = \{s_1^i, ..., s_{k_i}^i\}$ of strategies of all players $i \in I$. The *normal form* of a chess-like game (G, D, o, v_0) is defined as the pair (g, o).

For the backgammon-like games each strategy profile s uniquely determines a Markov chain, which assigns to each outcome $a \in A$ (that is, a terminal or an infinite play) a well defined limit probability $p(s, a)$. The payoff $u(i, s)$ of a player $i \in I$ in this situation s is defined as the expectation of the corresponding payoffs

$$u(i,s) = \sum_{a \in A} p(s,a)\, u(i,a).$$

A situation $s \in S$ is called a *Nash equilibrium* (NE) if for each player $i \in I$ and for each situation s' that may differ from s only in the coordinate i, the inequalities $u(i, s) \geq u(i, s')$ and $g(s) \succcurlyeq_{o_i} g(s')$ hold in case of the backgammon- and chess-like games, respectively; in other words, if no player $i \in I$ can profit replacing his/her strategy s^i in s by a new strategy s'^i, provided the n - 1 remaining players keep their strategies unchanged. Note that, since the preference o_i is strict, two situations s and s' are equally good for the player i if and only if the corresponding two outcomes coincide, that is, $g(s) =_{o_i} g(s')$ if and only if $g(s) = g(s')$.

3. The main example

The positional and normal forms of the game announced in the title of the paper are presented below by the figure and table, respectively.

4. Open ends

In the above example there are four players, $n = 4$, five terminals, $p = 5$, and the digraph contains only one dicycle. As we already mentioned, the following results concerning Nash-solvability are known: Every two-person chess-like game has a NE; see Section 3 of [7], Section 12 of [9], and/or Section 5 below. It was also shown in [7] that a chess-like game has a NE whenever condition (C) holds and $p \leq 2$. In [10] this result was strengthened: the bound $p \leq 2$ was replaced by $p \leq 3$.

Thus, the following three questions remain open: whether a chess-like game is Nash-solvable (i) when $n = 3$, or (ii) when $2 \leq p \leq 4$, or (iii) for any $n \geq 3$ and $p \geq 4$ provided condition (C) holds. It was conjectured in [26] that the answer to (iii) is negative. The corresponding example, if it exists, would strengthen simultaneously the example of Section 3 and the main example of [26]; see *Figure 1* and *Table 1* there.

Finally, it follows from the main result of [7] that an n-person chess-like game is Nash-solvable whenever each player controls a unique position. In the above example, the players 1, 2, 3, 4 control 2, 3, 2, 1 positions, respectively. It remains open, if there is a chess-like NE-free game in which each player controls, say, at most two positions.

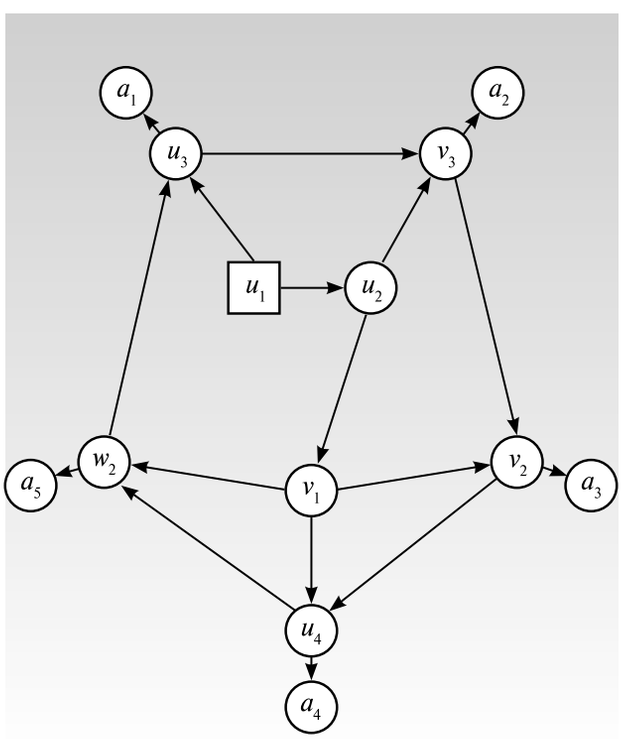

Fig. 1. The main example in the positional form

Table 1.

The main example in the normal form

		s_1^4								s_2^4							
		s_1^2	s_2^2	s_3^2	s_4^2	s_5^2	s_6^2	s_7^2	s_8^2	s_1^2	s_2^2	s_3^2	s_4^2	s_5^2	s_6^2	s_7^2	s_8^2
s_1^3	s_1^1	c^4	c^4	a_3^2	a_3^{23}	a_5^2	a_5^2	a_3^2	a_3^{23}	a_4^2	a_4^{23}	a_3^1	a_3^3	a_4^{24}	a_4^{234}	a_3^1	a_3^3
	s_2^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^{23}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_4^{234}	a_4^{234}	a_3^{13}	a_3^3
	s_3^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_5^{12}	a_3^{23}	a_4^2	a_4^{23}	a_4^{24}	a_3^3	a_4^{24}	a_4^{234}	a_4^{24}	a_3^3
	s_4^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^{23}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_4^{234}	a_4^{234}	a_3^{13}	a_3^3
	s_5^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_5^{12}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_5^{12}	a_4^{234}	a_5^{12}	a_3^3
	s_6^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^{23}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_4^{234}	a_4^{234}	a_3^{13}	a_3^3
s_2^3	s_1^1	a_1^3	a_1^3	a_3^{23}	a_3^{23}	a_5^{12}	a_5^{12}	a_3^2	a_3^{23}	a_4^{24}	a_4^{234}	a_3^1	a_3^3	a_4^{24}	a_4^{234}	a_3^1	a_3^3
	s_2^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_3^1	a_1^3	a_1^3	a_1^1	a_3^{23}	a_5^{12}	a_5^{12}	a_5^{12}	a_3^{23}	a_4^{24}	a_4^{234}	a_4^{24}	a_3^3	a_4^{24}	a_4^{234}	a_4^{24}	a_3^3
	s_4^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_5^1	a_1^3	a_1^3	a_1^1	a_3^{23}	a_5^{12}	a_5^{12}	a_5^{12}	a_3^{23}	a_1^1	a_4^{234}	a_1^1	a_3^{23}	a_5^{12}	a_4^{234}	a_5^{12}	a_3^{23}
	s_6^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
s_3^3	s_1^1	a_2^{23}	a_2^{23}	a_3^1	a_2^2	a_5^{12}	a_2^{23}	a_3^1	a_2^2	a_4^{124}	a_2^2	a_3^1	a_2^2	a_4^{124}	a_2^2	a_3^1	a_2^2
	s_2^1	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
	s_3^1	a_2^{23}	a_2^{23}	a_2^{23}	a_2^2	a_5^1	a_2^{23}	a_5^1	a_2^2	a_4^{14}	a_2^2	a_4^{14}	a_2^2	a_4^{14}	a_2^2	a_4^{14}	a_2^2
	s_4^1	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
	s_5^1	a_2^{23}	a_2^{23}	a_2^{23}	a_2^2	a_5^1	a_2^{23}	a_5^1	a_2^2	a_2^{23}	a_2^2	a_2^{23}	a_2^2	a_5^1	a_2^2	a_5^1	a_2^2
	s_6^1	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
s_4^3	s_1^1	a_1^3	a_2^{23}	a_3^2	a_2^2	a_5^{12}	a_2^{23}	a_3^2	a_2^2	a_4^{24}	a_2^2	a_3^1	a_2^2	a_4^{24}	a_2^2	a_3^1	a_2^2
	s_2^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_3^1	a_1^3	a_2^{23}	a_1^1	a_2^2	a_5^{12}	a_2^{23}	a_5^{12}	a_2^2	a_4^4	a_2^2	a_4^4	a_2^2	a_4^4	a_2^2	a_4^4	a_2^2
	s_4^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_5^1	a_1^3	a_2^{23}	a_1^1	a_2^2	a_5^{12}	a_2^{23}	a_5^{12}	a_2^2	a_1^1	a_2^2	a_1^1	a_2^2	a_5^{12}	a_2^2	a_5^{12}	a_2^2
	s_6^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1

Four players $I = \{1, 2, 3, 4\}$ make decisions in eight non-terminal positions u_1, v_1; u_2, v_2, w_2; u_3, v_3, and u_4, respectively. The subscript is the number of the player who controls the corresponding position.

The initial position is u_1. There are five terminal positions a_j: $j \in J = \{1, 2, 3, 4, 5\}$.

There is a unique dicycle c and, thus, the set of outcomes is $A = \{a_1, a_2, a_3, a_4, a_5; c\}$.

The game has no NE whenever the preferences o_i of the players $i \in I$ over the set of outcomes A agree with the following partial orders:

$O_1 : a_2 \succ_{o_1} a_4 \succ_{o_1} a_3 \succ_{o_1} a_1 \succ_{o_1} a_5$;

$O_2 : min_{o_2}(a_1, c) \succ_{o_2} a_3 \succ_{o_2} max_{o_2}(a_4, a_5) \succ_{o_2} min_{o_2}(a_4, a_5) \succ_{o_2} a_2$;

$O_3 : min_{o_3}(a_5, c) \succ_{o_3} a_1 \succ_{o_3} a_2 \succ_{o_3} max_{o_3}(a_3, a_4)$;

$O_4 : min_{o_4}(a_1, a_2, a_3, a_5) \succ_{o_4} a_4 \succ_{o_4} c$.

The game form $g: S \to A$, in which $S = S_1 \times S_2 \times S_3 \times S_4$ and $A = \{a_1, a_2, a_3, a_4, a_5; c\}$, is given by the four-dimensional table of size $6 \times 8 \times 4 \times 2$.

Player 1 has six strategies:

s_1^1: (u_1, u_2), (v_1, v_2), s_2^1: (u_1, u_3), (v_1, v_2),

s_3^1: (u_1, u_2), (v_1, u_4), s_4^1: (u_1, u_3), (v_1, u_4),

s_5^1: (u_1, u_2), (v_1, w_2), s_6^1: (u_1, u_3), (v_1, w_2);

player 2 has eight strategies:

s_1^2: (u_2, v_1), (v_2, u_4), (w_2, u_3), s_2^2: (u_2, v_3), (v_2, u_4), (w_2, u_3),

s_3^2: (u_2, v_1), (v_2, a_3), (w_2, u_3), s_4^2: (u_2, v_3), (v_2, a_3), (w_2, u_3),

s_5^2: (u_2, v_1), (v_2, u_4), (w_2, a_5), s_6^2: (u_2, v_3), (v_2, u_4), (w_2, a_5),

s_7^2: (u_2, v_1), (v_2, a_3), (w_2, a_5), s_8^2: (u_2, v_3), (v_2, a_3), (w_2, a_5);

player 3 has four strategies:

s_1^3: (u_3, v_3), (v_3, v_2), s_2^3: (u_3, a_1), (v_3, v_2),

s_3^3: (u_3, v_3), (v_3, a_2), s_4^3: (u_3, a_1), (v_3, a_2);

finally, player 4 has two strategies: $s_1^4 : (u_4, w_2)$, $s_2^4 : (u_4, a_4)$,

It is not difficult (although time consuming) to verify that the game has no NE for any preference profile specified above.

For every situation

$$s = \left(s_{l_1}^1, s_{l_2}^2, s_{l_3}^3, s_{l_4}^4\right) \in S_1 \times S_2 \times S_3 \times S_4$$

the outcome $g(s)$, which is either a terminal a_j or the dicycle c, is shown in the entry (l_1, l_2, l_3, l_4) of the table. The upper indices indicate the players who can improve the situation s. Thus, a situation s is a NE if and only if the corresponding outcome has no upper indices. Since the table contains no such situation, the considered game has no NE.

5. Two-person chess-like games are Nash-solvable

The proof can be found in [7] (and also in [9]; see the last section of each paper); yet, since the proof is very short, we will repeat it here for convenience of the reader. It is based on the following important property of the two-person game forms, which seems not to be extendable for $n > 2$.

A two-person game form g is called (i) *Nash-solvable*, (ii) *zero-sum-solvable*, and (iii) ± 1-*solvable* if the corresponding game (g, u) has at least one NE (i) for every payoff $u = (u_1, u_2)$; (ii) for every payoff $u = (u_1, u_2)$ such that $u_1(a) + u_2(a) = 0$ for each outcome $a \in A$; (iii) for every payoff $u = (u_1, u_2)$ such that $u_1(a) + u_2(a) = 0$ for each outcome $a \in A$ and both u_1 and u_2 take only values +1 or -1.

In fact, all three above properties of a game form are equivalent. For (ii) and (iii) this was shown in 1970 by Edmonds and Fulkerson [12] and independently in [19]. Then, the list was extended by statement (i) in [20]; see also [21], where it also was shown that a similar statement fails for the three-person game forms.

Thus, it is sufficient to prove ± 1-solvability, rather than Nash-solvability, of the two-person chess-like games. Hence, we can assume that each outcome $a \in A = V_T \cup \{c\}$ is either winning for player 1 and losing for player 2, or vice versa. Without any loss of generality, assume that c is winning for 1.

Then, let $V_T = V_T^1 \cup V_T^2$ be the partition of all terminals into outcomes winning for players 1 and 2, respectively. Furthermore, let $V^2 \subseteq V$ denote the set of all positions from which player 2 can enforce V_T^2; in particular, $V_T^2 \subseteq V^2$. Finally, let us set $V^1 = V \setminus V^2$; in particular, $V_T^1 \subseteq V^1$. By the above definitions, in every position $v \in V_1 \cap V^1$ player 1 can stay out of V^2, that is, (s)he has a move (v, v') such that $v' \in V^1$. Let us fix a strategy s_0^1 that chooses such a move in each position $v \in V_1 \cap V^1$ and any move in $v \in V_1 \cap V^2$. Then, for any $s^2 \in S_2$, the outcome $g(s_0^1, s^2)$ is winning for player 1 whenever the initial position v_0 is in V^1. Indeed, either $g(s_0^1, s_2) \in V_T^1$, or $g(s_0^1, s^2) = c$; in both cases player 1 wins. Thus, player 1 wins when $v_0 \in V^1$ and player 2 wins when $v_0 \in V^2$; in each case a saddle point exists.

6. Related results on Nash-solvability

In the next two subsections we recall two large families of games with perfect information that are known to be Nash-solvable in pure stationary uniformly optimal strategies.

6.1. Acyclyc *n*-person backgammon-like games with perfect information

In 1950 Nash introduced his concept of equilibrium for the normal form n-person games [35, 36]. Soon after, Kuhn [30, 31] and Gale [16] suggested the so-called backward induction procedure and proved that any finite acyclic chess-like game with perfect information has a NE in pure stationary strategies; moreover, the obtained NE is *subgame perfect*, that is, the same strategy profile is a NE with respect to any initial position. The authors restricted themselves to the chess-like games on finite arborescence (directed trees) but, in fact, backward induction can be easily extended to the backgammon-like games on the finite digraphs without dicycles. Yet, acyclicity is a crucial assumption and cannot be waved.

For any integer $k \geq 2$ let us introduce a digraph G_k that consists of k terminals $a_1, \ldots, a_k$, the directed

k-cycle C_k on the k non-terminal vertices $v_1, ..., v_k$, and the perfect matching (v_j, a_j), $j = 1, ..., k$ between these vertices and the terminals.

The existence of a subgame perfect NE fails already for $k = 2$ [1]. Let players 1 and 2 control vertices v_1 and v_2 and have the preferences, $(c > a_1 > a_2)$ and $(a_1 > a_2 > c)$, respectively. It is easy to verify that a NE exists for any given initial position, v_1 or v_2, but no strategy profile is a NE with respect to both simultaneously. Let us notice that the preferences are not opposite (both players prefer a_1 to a_2), while c is the worst outcome for player 2 and the best one for 1.

A similar example exists even if in addition we require (C): the dicycle is worse than each terminal for both players. Consider digraph G_6 in which players 1 and 2 control the odd and even positions (a_1, a_3, a_5 and a_2, a_4, a_6) respectively. It was shown in [3] that there exists no subgame perfect NE whenever

$o_1: a_6 \succ_{o_1} a_5 \succ_{o_1} a_2 \succ_{o_1} a_1 \succ_{o_1} a_3 \succ_{o_1} a_4 \succ_{o_1} c$;

$o_2 \in O_2$: $\{a_3 \succ_{o_2} a_2 \succ_{o_2} a_6 \succ_{o_2} a_4 \succ_{o_2} a_5 \succ_{o_2} c$ and $a_6 \succ_{o_2} a_1 \succ_{o_2} c\}$.

Let us note that there exists no such example for G_4. Yet, for a three-person game, the following similar example was constructed in [37] and [7], independently; see also [14] and [3].

The players 1, 2, 3 control, respectively, the positions v_1, v_2, v_3 of G_3 and have the preferences:

$O_1: a_2 \succ_{o_1} a_1 \succ_{o_1} a_3 \succ_{o_1} c$;

$O_2: a_3 \succ_{o_2} a_2 \succ_{o_2} a_1 \succ_{o_2} c$;

$O_3: a_1 \succ_{o_3} a_3 \succ_{o_3} a_2 \succ_{o_3} c$.

In other words, for all players: c is the worst outcome, in accordance with (C), it is better if the previous player terminates the play, still better to terminate himself, and it is the best if the next player terminates.

In [6], these results were strengthen as follows. It was demonstrated that a subgame perfect NE may fail to exist not only in the pure but even in the mixed strategies. The corresponding examples are based on the same positional game forms, G_6 for $n = 2$ and G_3 for $n = 3$, but the above preference profiles are replaced by some carefully chosen payoffs, which agree with the corresponding preferences.

The above examples imply that, for any $n \geq 2$, an n-person *backgammon-like* game, even with a fixed initial position, may have no NE. Given a chess-like game $(G = (V, E), D, o)$, in which no initial position is fixed yet, add to it a new position v_0 and the move (v_0, v) from v_0 to each non-terminal position $v \in V \setminus V_T$. Furthermore, assign to (v_0, v) a non-negative probability $p_v \geq 0$ such that $\sum_{v \in V} p_v = 1$. Denote by (G_k', D, o, v_0) the obtained backgammon-like game form, in which v_0 is the initial position; see *Fig. 1* in [3]. It was shown in [3] that

✦ (i) if s is a subgame perfect NE in (G, D, o) then s is a NE in (G', D, o, v_0) for any p_v;

✦ (ii) if s is a NE in (G', D, o, v_0) and $p_v > 0 \; \forall v \in V \setminus V_T$, then s is a subgame perfect NE in (G_k, D, o).

These results imply that Nash-solvability of (G', D, v_0) is equivalent with subgame perfect Nash-solvability of (G, D). As we know, the latter property may fail for $G = G_k$ for any $n \geq 2$. Thus, for any $n \geq 2$, an n-person backgammon-like game (G_k', D, v_0, o), in which v_0 is a unique position of chance, may have no NE.

6.2. Two-person zero-sum games with perfect information

According to the previous subsection, in the presence of dicycles, backward induction fails, in general. Yet, it can be modified (and thus saved) in case of the two-person zero-sum games (that is, when $I = \{1, 2\}$ and $u(1, a) + u(2, a) = 0$ for any outcome $a \in A$, or when two preferences o_1 and o_2 of the players 1 and 2 over A are opposite).

For example, the recent paper [18] shows how to solve by backward induction a two-person zero-sum game that is «acyclic», but the players can pass; in other words, the corresponding digraph contains a loop at each vertex, but no other dicycles.

A general linear time algorithm solving any two-person zero-sum chess-like game, by a modified backward induction, was suggested in [2] and independently in [3]. In contrast, no polynomial algorithm is known for the two-person zero-sum backgammon-like games [11]. However, it is well-known that subgame perfect saddle points in stationary strategies exist in this case and even in much more general cases considered below.

In fact, studying two-person zero-sum chess-like games began long before the backward induction was suggested in early fifties by [30, 31, 16]. Zermelo gave his seminal talk on solvablity of chess in pure strategies [40] as early as in 1912. Later, König [29] and Kalman [28] strengthen this result showing that there exist pure stationary uniformly optimal strategies producing a subgame perfect saddle point, in any two-person zero-sum chess-like game.

The chess-like and backgammon-like games, considered in this note, by the definition, are «transition free». There is a much more general class: *stochastic games with perfect information* in which a *transition payoff* $r_i(u, v)$ is defined for every move (u, v) and for each play-

er $i \in I$. Gillette, in his seminal paper [17], introduced the *mean* (*or average*) effective payoff for these games and proved the existence of subgame perfect saddle point in the uniformly optimal stationary strategies for the two-person zero-sum case. The proof is pretty complicated. It is based on the Tauberian theory and, in particular, on the Hardy-Littlewood theorem [27]. (In [17], the conditions of this theorem were not accurately verified and the flaw was corrected in twelve years by Liggett and Lippman in [32].)

Stochastic games with perfect information can be viewed as *backgammon-like* games with transition payoffs. More precisely, these two classes are polynomially equivalent [4]. Interestingly, the corresponding two-person zero-sum *chess-like* games (with transition payoffs but without random moves) so-called *cyclic mean-payoff games*, appeared only 20-30 years later, introduced for the complete bipartite digraphs by Moulin [33, 34], for any bipartite digraphs by Ehrenfeucht and Mycielski [13], and for arbitrary digraphs by Gurvich, Karzanov, and Khachiyan [25]. Again, the existence of a saddle point in the pure stationary uniformly optimal strategies was proven for the two-person zero-sum case.

This result cannot be extended to the non-zero-sum case. In [22], a cyclic mean payoff two-person NE-free game was constructed on the complete bipartite 3×3 digraph with symmetric payoffs. (The corresponding normal form game is a 27×27 bimatrix.) It was shown in [23] that this example is, in a sense, minimal, namely, a NE always exists for the games on the complete $(2 \times l)$ bipartite digraphs.

A general family of the so-called *k-total effective payoffs* was recently introduced in [5] for any nonnegative integer k such that the 0-total one is the mean effective payoff, while the 1-total one is the total effective payoff introduced earlier by Thuijsman and Vrieze in [38, 39]. The existence of a saddle point in uniformly optimal pure stationary strategies for the two-person zero-sum chess-like games with the k-total effective payoff was proven for all k in [5]. If this result can be extended to the backgammon-like games is an open problem. Yet, for $k \leq 1$ the answer is positive. As was already mentioned, for $k = 0$ it was proven long ago. For $k = 1$ the result was first obtained in [38, 39], see also [5].

However, it cannot be extended to the non-zero-sum case. In particular, a NE (in pure stationary strategies) may fail to exist already in a two-person chess-like game. For $k = 0$ the example was given in [22]. Furthermore, in [5], a simple embedding of the $(k$-1)-total payoff games into the k-total ones was constructed. Thus, the example of [22] works for all k.

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ИГРА ЧЕТЫРЕХ ЛИЦ С ПОЛНОЙ ИНФОРМАЦИЕЙ И БЕЗ СЛУЧАЙНЫХ ХОДОВ, НЕ ИМЕЮЩАЯ СИТУАЦИЙ РАВНОВЕСИЯ НЭША В ЧИСТЫХ СТАЦИОНАРНЫХ СТРАТЕГИЯХ

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В статье рассматривается пример конечной позиционной игры с полной информацией и без случайных ходов (так называемая игра шахматного типа), которая не имеет равновесий Нэша в чистых стационарных стратегиях. В этом примере число игроков n равно 4, число терминальных позиций p равно 5; при этом граф игры имеет всего один ориентированный цикл.

С другой стороны, известно, что игра шахматного типа имеет равновесие Нэша в чистых стационарных стратегиях, если выполнено хотя бы одно из следующих трех условий: либо (A) $n \leq 2$; либо (B) $p \leq 3$ и (C) любой игрок предпочитает любую терминальную позицию любой бесконечной партии; либо (D) каждый из n игроков контролирует всего одну позицию; либо (E) граф не имеет ориентированных циклов.

Остается открытым вопрос, существует ли игра шахматного типа, имеющая хотя бы один ориентированный цикл и не имеющая равновесий Нэша в следующих четырех случаях: (A') $n = 3$; (B') $2 \leq p \leq 4$; (C') $n > 2$, $p > 3$ и условие (C) выполняется; (D') каждый из n игроков контролирует не более двух позиций.

В нашем примере $n = 4$, $p = 5$, условие (C) не выполнено и один из игроков контролирует три позиции.

Ключевые слова: позиционная игра, стохастическая игра, игра шахматного типа, полная информация, случайный ход, равновесие Нэша, терминал, ориентированный цикл.

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