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Application of measures of heavy-tailedness in problems for analysis of financial time series

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Abstract

An important feature when working with financial data is the fact that the residuals of GARCH-models often have fatter tails than the tails of a normal distribution due to the large number of "outliers" in the data. This requires more detailed study. Kurtosis and quantile-based measure of heavy-tailedness were analyzed and compared in the article in relation to the problem of choosing the GARCH(1,1)-model specification. The data of indices of the Moscow Exchange were considered for the period from April 01, 2019 to February 22, 2022. Kurtosis values ranged from 3 to 52. Empirical data showed that kurtosis was very sensitive to "outliers" in the data, which made it difficult to make assumptions about the distribution of model residuals. The approach considered in this paper based on the heavy-tailedness measure made it possible to justify the choice of degrees of freedom of the t-distribution for the model residuals to explain the fat tails in financial data. It was found that GARCH(1,1)-models with $t(5)$-distribution in the residuals are common.

Keywords: GARCH, kurtosis, quantile-based measure of heavy-tailedness, *t*-distribution of residuals, degrees of freedom, fat tails

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Introduction

As is known, many financial time series are characterized by certain regularities: asset returns are weakly stationary, volatility clustering is observed, distribution normality is rejected in favor of a distribution with thick tails, etc. [1]. To describe and forecast the processes with such properties, wide use is made of the class of models with conditional heteroscedasticity (ARCH, GARCH models) proposed by Angle [2] and Bollerslev [3] and their modifications. An important feature when working with financial data which we would like to consider in detail in this article is the fact that the residuals of ARCH/GARCH models have fatter tails than the tails of a normal distribution due to the large number of "outliers" in the data, and this fact requires more detailed study. To account for fat tails in econometric practice, several alternative distributions had been proposed: Student's *t*-distribution [3, 4], generalized error distribution (GED) [5, 6], Student's skew *t*-distribution [7], etc. Note that the possibility of choosing the Student's *t*-distribution and GED-distribution for estimating the GARCH model is implemented in econometric packages (for example, Stata16). This is of practical interest in substantiating the choice of the appropriate distribution in modeling and forecasting. The proposed distributions differ in properties; therefore, these distributions will not equally well characterize the "thickness" of the tails of the distribution. Thus, the *research problem* is how to choose the type of distribution that best characterizes the heavy-tailed distribution. The correct specification of the GARCH model, taking into account heavy tails, allows us to get more accurate forecasts of returns and the maximum profit for investors. This fact determines the *relevance* of the study.

The main *goal* of this paper is to analyze the behavior of the quantile-based measure of tailedness in relation to the choice of degrees of freedom of the Student's *t*-distribution in the residuals of the GARCH model. Note that the measures of heavy-tailedness are widely discussed in foreign literature and are an alternative approach to choosing the number of degrees of freedom of the *t*-distribution. Let us check how applicable these measures are in econometric practice for analyzing financial data and compare them with the classical approach of choosing the degree of freedom of the *t*-distribution based on a comparison of maximum likelihood estimates.

1. Measurement of heavy-tailedness

In this section, we will analyze what approaches are used for measuring the "thickness" of the tail of the distribution. The heavy-tailedness of a distribution for a random variable (r.v.) X is usually understood as

$$P(|X| > x) \sim \frac{C}{x^{\xi}}, \qquad (1)$$

where $C, \xi > 0$ are the constants, $f(x) \sim g(x)$ means:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

The parameter ξ is commonly called the tail index of the distribution X. It characterizes the decay rate of the tails of the power-law distribution (1) and the probability of observing extreme values of the r.v. As the probability mass in the tails increases, the tail index parameters decrease, and vice versa [8].

Note that in the literature, heavy-tailed distributions are divided into three subclasses: fat-tailed distributions, long-tailed distributions, and subexponential distributions [9, 10]. A fat tail distribution exhibits more skewness or kurtosis than a normal distribution. The terms "fat tail" and "heavy tail" are often used as synonyms in financial analysis papers. In our work, we will use the term "heavy tail", and we will consider the "fat tails" of the distribution as a special case of "heavy tails." In practice, we have the question of how to measure the heavy-tailedness, and how to assess the degree of "heaviness" of the tail of the distribution. There are parametric and nonparametric approaches to estimating the tail index [11]. In our article, we will analyze the "heavy-tailedness" in the context of time series modeling based on GARCH models. Kurtosis is one of the measures used to detect "outliers" in time series. In 1905 Pearson introduced the concept of kurtosis through 4th order moments:

$$K = \frac{\mu_4}{\sigma^4}, \tag{2}$$

where μ_4 is the central moment of the 4th order;

σ^4 is the square of the variance.

All distributions were classified as plateaucurtic, mesocurtic or leptokurtic with respect to normal [12]. For a normal distribution $K = 3$, in connection with which an excess kurtosis (modified kurtosis indicator) is often used:

$$E_k = \frac{\mu_4}{\sigma^4} - 3.$$

Kurtosis K in the form (2) will be used further in the article. Fat tails (as a special case of heavy tails) are characterized by excessive kurtosis $K > 3$, and the distribution is called leptokurtic [13].

In this paper, we will compare and explore the residuals of GARCH(1,1) models as often used in econometric practice [14]. Recall the definition. The process ε_t follows the generalized autoregressive conditional heteroscedasticity or GARCH(1,1)-model if $\varepsilon_t = \sigma_t z_t$, $t = 1, 2, \ldots$, where $z_t \sim N(0,1)$ is independent normally distributed random variables, and the conditional variance of the process has the form:

$$\sigma_t^2 = \omega + \gamma\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \tag{3}$$

Note that in practice, model (3) only partly explains the fat tails, and it is necessary to refine the specification of the distribution of residuals. Student's t-distribution is often used as an alternative to the normal distribution [3, 4]. The standardized Student's t-distribution with zero mean and unit variance has a density:

$$f(z_t, v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{1}{2}v\right)\sqrt{\pi(v-1)}}\left[1 + \frac{z_t^2}{v-2}\right]^{-\frac{v+1}{2}},$$

where $\Gamma(.)$ is Euler gamma function;

$v > 2$ is number of degrees of freedom.

The kurtosis of the distribution z_t is

$$K = 3\frac{v-2}{v-4}, \; v > 4.$$

The kurtosis of the errors ε_t is

$$K_u = 3\,\frac{v-2}{v-4}\,\frac{E(\sigma_t^4)}{\left[E(\sigma_t^2)\right]^2}. \tag{4}$$

The factor containing v in expression (4) makes it possible to take into account the excess kurtosis [1]. It can be seen from formula (4) that the kurtosis of the t-distribution depends on the degree of freedom and the degree of volatility of the process. Thus, by varying the degrees of freedom of the t-distribution, we can obtain different values of kurtosis and different degrees of heavy-tailedness. However, kurtosis based on 4th-order moments is very sensitive to outliers, and therefore, in the presence of outliers, can lead to false conclusions about the nature of the distribution of the residuals.

The question of whether kurtosis measures heavy-tailedness and how to determine which distribution has a heavier tail based on kurtosis is debatable. If, for example, we look at kurtosis as an average of outliers, then large kurtosis indicates large heavy tails [15]. Some authors describe kurtosis as a measure of both "pointiness" and "tail thickness" [16]. In general, there are three approaches to comparing the heaviness of the tails of distributions: the usual kurtosis K, the measures of "heaviness" of the tails, and the ratio of interquantile intervals. Consider one of the approaches – a measure of heavy-tailedness based on quantiles, in the form:

$$K_{\alpha}\left(y_t\right)=\frac{Q_{1-\alpha}(y_t)-Q_{\alpha}(y_t)}{Q_{1-\tau}(y_t)-Q_{\tau}(y_t)}, \qquad (5)$$

where $Q_{\theta}(y_t)$ is θ-quantile y_t, $0<\alpha<\tau<0.5$ [16].

Following article [17], we choose $\tau=0.25$ and $\alpha=0.01$, $\alpha=0.05$. It is believed that the quantile measure (5) is free from assumptions about distributions and from kurtosis values. Therefore it is resistant to misclassifications of distributions and can be used to compare distributions. We calculate and study the behavior of kurtosis (2) and measure of heavy-tailedness (5) for the normal distribution and Student's t-distribution, as a commonly used distribution in econometric practice to account for fat tails, and compare it with the distributions of empirical returns data, which will allow us to make an assumption about the number degrees of freedom t-distribution for empirical data when specifying GARCH(1,1)-models. We will compare this with the distributions of empirical data of returns, which will allow us to make an assumption about the degrees of freedom t-distribution for empirical data when specifying GARCH(1,1)-models.

2. Analysis of heavy tails of distribution in Russian studies

In this section, we will analyze how heavy tails of distributions are taken into account in practice when forecasting financial series of returns in recent studies. To account for fat tails in econometric practice, Student's t-distribution and its variations are often used. Shvedov noted the importance of using t-distribution for MLE (maximum likelihood estimation) estimates in the case of outliers in the data. The author compared the EM-algorithm (expectation-maximization algorithm) and LSM (least-squares method) for linear regression model estimates on generated data with different distributions of errors [18]. Balaev considered and compared the two-dimensional t-distribution with a vector and a scalar of degrees of freedom, the generalized error distribution and the Gram-Charlier distribution according to the daily closing prices of stock indices of various countries: S&P 500, FTSE 100, CAC 40, DAX, Hang Seng, Nikkei during the period November 26, 1990 – November 18, 2012. The author noted that the distributions of all considered returns have heavy tails. The kurtosis coefficient varied from 5.21 to 9.45. It was found that the model based on t-distribution with a vector of degrees of freedom was more preferable [19]. Works of Fantazzini were of a survey nature and were devoted to modeling multivariate distributions based on copula functions. The paper introduced the concepts of upper and lower tail dependences for random variables with a certain probability of outliers and considers Student's copula functions [20]. Balazs in his work analyzed the influence of external sources of information (news and trading volumes) on the volatility of securities using GARCH(1,1)-models. The author noted that the hypothesis of normality was rejected for most of the securities under consideration (returns on shares of 19 companies from the FTSE100 list for the period July 01, 2005 – July 01, 2008) [21]. Some authors used a class of special models Go-GARCH, GJR-GARC, which allow one to estimate the degrees of freedom of the t-distribution along with other model parameters [22, 23]. Lakshina simulated returns with further calculation of the dynamic hedge ratio for eight shares of Russian companies traded on the RTS for the period January 01, 2007 – October 01, 2014. Kurtosis ranged from 18 to 42. Based on the GO-GARCH model, it was calculated that the residuals were Student's distribution with 2 degrees of freedom.

Note that the authors of the papers under consideration used the kurtosis as an "indicator" of heavy tails, and the question of choosing the degrees of freedom t-distribution usually remained outside the scope of such studies. It is also known that the inclusion of the degrees of freedom t-distribution in the arguments of the likelihood function in the MLE is not always correct [24]. Thus, some alternative characteristics of the "heavy-tailedness" are needed, which are easily implemented in practice. In our work, we decided to make an attempt to fill this gap.

3. Analysis of the measure of heavy-tailedness and kurtosis for Student's distribution on generated data

Consider the behavior of the measure of heavy-tailedness in the form (5) and kurtosis (2) for Student's $t(v)$-distribution for different degrees of freedom v. Using the Monte Carlo method, we will generate 5000 repetitions over $N = 200$, 750 and 1000 observations, calculate and compare the kurtosis K and measures of heavy-tailedness K_{01}, K_{05} for quantiles $\alpha = 0.01$, $\alpha = 0.05$, respectively: the interval of variation from minimum to maximum (min-max) and average (mean). The simulation results are shown in *Table 1.* Random variables were generated in the Stata16 package. The generation of pseudo-random numbers was implemented on the basis of the algorithm proposed in the work [25]. Due to the properties of t-distribution, we considered degrees of freedom from 3 to 10. Note that theoretical kurtosis exists for $v > 4$.

The values of the measures from *Table 1* will be further used for comparison with the measures of heavy-tailedness of the analyzed returns of the Moscow Exchange indices for further specification of the GARCH(1,1)-model in choosing the assumption about the distribution of residuals.

Table 1.

Measures of heavy-tailedness (5) and kurtosis (2) for $t(v)$-Student's distribution and normal distribution

	$K_{01min-max}$	K_{01mean}	$K_{05min-max}$	K_{05mean}	$K_{min-max}$	K_{mean}
$N = 1000$						
$t(3)$	4.8–7.7	5.96	2.71–3.62	3.08	5.7–464.07	29.883
$t(4)$	3.9–7.08	5.085	2.51–3.37	2.875	4.09–304.55	12.653
$t(5)$	3.76–5.61	4.645	2.41–3.17	2.772	3.88–115.6	7.550
$t(6)$	3.71–5.35	4.388	2.35–3.13	2.708	3.5–127.91	6.166
$t(7)$	3.57–5.02	4.224	2.29–3.11	2.662	3.35–26.54	4.679
$t(8)$	3.45–4.97	4.103	2.26–3.13	2.633	3.30–16.28	4.486
$t(9)$	3.37–4.76	4.016	2.28–3.13	2.609	3.04–13.85	4.186
$t(10)$	3.38–4.68	3.947	2.29–3.11	2.589	3.01–13.83	4.013
$N(0,1)$	3.01–4.07	3.467	2.18–2.74	2.442	2.57–3.84	3.004
$N = 750$						
$t(3)$	4.37–8.94	5.989	2.57–3.75	3.084	4.28–572.55	27.596
$t(4)$	3.81–7.52	5.080	2.39–3.49	2.881	3.34–614.12	12.881
$t(5)$	3.45–6.49	4.649	2.31–3.37	2.771	3.31–541.35	7.391
$t(6)$	3.39–5.82	4.397	2.27–3.25	2.707	3.26–147.19	5.672
$t(7)$	3.34–5.31	4.216	2.27–3.21	2.663	3.06–48.49	4.909
$t(8)$	3.21–5.42	4.105	2.23–3.25	2.631	3.02–55.67	4.411
$t(9)$	3.21–5.21	4.013	2.15–3.07	2.606	2.89–23.87	4.152
$t(10)$	3.12–5.11	3.951	2.18–3.20	2.591	2.89–24.09	3.975
$N(0,1)$	2.82–4.37	3.465	2.13–2.84	2.442	2.49–4.36	2.995

	$K_{01min-max}$	K_{01mean}	$K_{05min-max}$	K_{05mean}	$K_{min-max}$	K_{mean}
$N = 200$						
$t(3)$	3.87–12.45	6.259	2.25–4.31	3.108	3.28–137.78	13.783
$t(4)$	3.42–9.54	5.237	2.21–3.74	2.897	3.12–141.47	8.775
$t(5)$	3.17–7.75	4.756	2.14–3.73	2.790	2.78–62.72	6.123
$t(6)$	2.81–6.46	4.459	1.97–3.64	2.714	2.71–65.19	5.200
$t(7)$	2.97–7.72	4.307	1.99–3.75	2.670	2.52–21.59	4.648
$t(8)$	2.96–6.31	4.155	2.01–3.40	2.633	2.61–26.43	4.266
$t(9)$	2.96–5.56	4.058	2.05–3.62	2.611	2.51–26.48	4.086
$t(10)$	2.73–6.24	3.987	2.00–3.51	2.588	2.31–30.24	3.943
$N(0,1)$	2.72–4.92	3.504	2.01–3.23	2.446	2.32–4.28	3.007

Behavior of measures of heavy-tailedness – key findings:

- Measures of heavy-tailedness $K_{01,}$ K_{05} (5) based on quantile estimates are more robust to outliers than kurtosis calculated from distribution moment estimates. Measures lie in fixed ranges depending on the degree of freedom, while the kurtosis varies greatly for all considered degrees of freedom v from 3 to 10. It is rather difficult to make assumptions about the degrees of freedom of the t-distribution over the kurtosis values.
- Measures of heavy-tailedness K_{01} for quantile 0.01 are more informative, as K_{05} measures have many overlapping intervals. In what follows, to justify the choice of the degrees of freedom v of the t-distribution, we will use the K_{01} measure, and the K_{05} measure will be used to control the range of the measure using empirical data.
- The resulting intervals of variation of measures of heavy-tailedness for theoretical distributions will be used in further work to compare with measures of heavy-tailedness based on empirical data and substantiate the assumption regarding the degrees of freedom of the t-distribution of residuals in GARCH(1,1)-model.
- It can be assumed that the measures of heavy-tailedness comparison approach works well on large samples (N = 700 and more). Note that in [17], measures of heavy-tailedness were also calculated based on interfractile range (α = 0.125), which requires a sample of at least 1000 observations. This can be attributed to the disadvantages of this approach, since economic data are not always expressed in long time series.

4. Measures of heavy-tailedness and kurtosises for Moscow Exchange indices

In this paper, we examined the returns (logarithmic differences) of the MOEX Index: major and sectoral indices for the period from April 01, 2019 to February 23, 2022 (732 trading days) [26]. The opening and closing prices (the price of the first and last transaction on the trading day) were studied. Graph of returns of the oil and gas index (closing price) is shown in *Fig. 1*. The series of returns has a constant zero mean value, and clustering volatility is observed. The period $t < 200$ (April 01, 2019 – February 20, 2020) is marked as a period of low volatility. The returns dynamics of other indices behave in a similar way. Periods of high volatility, as a rule, are characterized by abnormally high values (in absolute value) of returns, which leads to high values of kurtosis and the appearance of "fat" distribution tails. Note that indices with different kurtosis values were taken for further analysis.

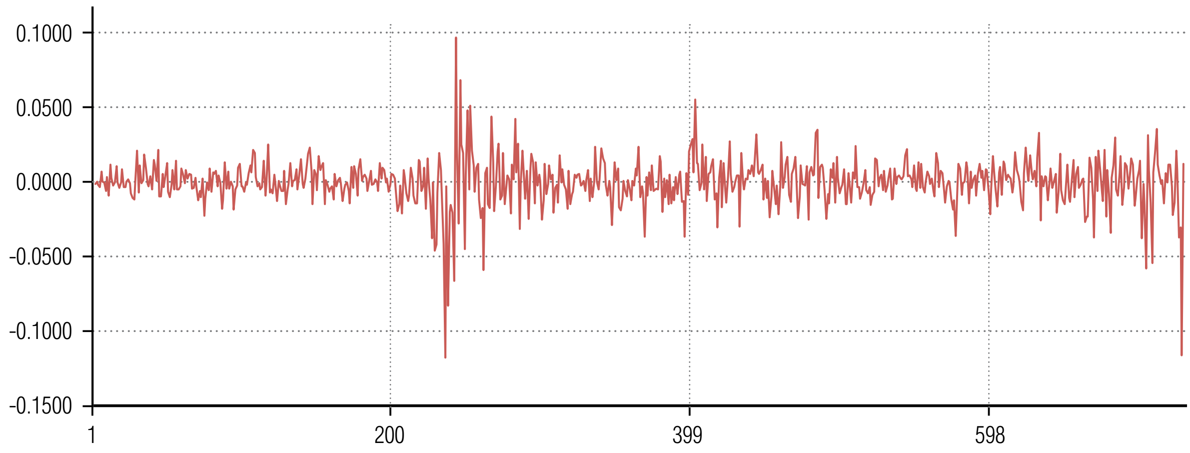

Fig. 1. Logarithmic difference of Oil and Gas Index (Closing Price) (*y*-axis) for the period from April 01, 2019 to February 23, 2022 (*x*-axis).

Let us introduce the notation in *Tables 2, 3*:

- Period 1: April 01, 2019 – February 23, 2022 ($N = 732$), period 2: April 01, 2019 – February 20, 2020 ($N = 200$);
- $K1$, $K2$ are kurtoses (2) of the returns of the Moscow Exchange indices for period 1 and 2, respectively;
- $K1_{01}$, $K1_{05}$ and $K2_{01}$, $K2_{05}$ are measures of heavy-tailedness (5) for periods 1 and 2 for 0.01 and 0.05 quantiles, respectively.

Tables 2, 3 contain the values of kurtoses $K1$ and $K2$ of index returns for two periods: period 1 has a pronounced clustering volatility, period 2 – the period of low volatility. Note that period 1 is characterized by a significant difference in the value of the kurtosis between the opening and closing prices by almost two times, and the kurtosis varies from 6.77 to 51.87. The opening price (variables with index 2 in *Tables 2, 3*) and the closing price (variables with index 1 in *Tables 2, 3*) as different variables were used further for modeling. It is obvious that the value of kurtoses of indicators for period 1 indicates that the assumption of normality of residuals in GARCH(1,1) cannot be used. In period 2, the differences in the value of kurtoses for the opening and closing prices are insignificant, and for some indices they coincide with the kurtosis of the normal distribution ($K = 3$) (*Tables 2, 3*). Thus, further analysis and comparison of estimates of GARCH(1,1)-models in the work was carried out for period 1.

It should be noted that the measures of heavy-tailedness $K2_{01}$ (5) of most of the analyzed indices (85%) calculated for period 2 fall within the intervals of varying measures for a normal distribution ($N = 200$): 2.72–4.92 (*Table 1*). The measures of heavy-tailedness $K2_{01}$ (5) of some variables calculated for period 2 are given in *Table 2*. Thus, it can be assumed that GARCH(1,1)-models, assuming normal residuals, will be the best model for a period with low volatility.

Assumptions about the degrees of freedom of the $t(v)$-distribution of residuals for further specification of GARCH(1,1)-models by comparing the calculated measures of heavy-tailedness with the value of measures of theoretical distributions (*Table 1*) based on the calculated measures were made for period 1. For example, index blue1 has measure of heavy-tailedness $K1_{01} = 6{,}009$ (*Table 2*). This measure corresponds to the variation interval $K01_{min-max}$ ($N = 750$) for $t(3)$: 4.37–8.94; $t(4)$: 3.81–7.52; $t(5)$: 3.45–6.49 (*Table 1*). Therefore, $t(3)$–$t(5)$ will be the assumed distributions of the residuals when estimating the GARCH(1,1)-models. We will consider how well the considered measure of heavy-tailedness allows us to correctly specify GARCH(1,1). From *Tables 2, 3* it can be seen that mea-

Table 2.

Values of kurtoses (2) and measures of heavy-tailedness (5) of index returns (logarithmic differences), *main equity* indices

MOEX Index		Period 1				Period 2		
		$K1$	$K1_{01}$	$K1_{05}$	Estimated $t(v)$ for residuals	$K2$	$K2_{01}$	$K2_{05}$
blue1	MOEX Blue Chip Index	15.194	6.009	2.946	$t(3)-t(5)$	3.597	3.991	2.287
blue2		33.434	6.225	2.966	$t(3)-t(5)$	3.579	4.541	2.509
imoex1	MOEX Russia Index	15.596	6.347	3.019	$t(3)-t(5)$	3.406	3.792	2.347
imoex2		32.844	6.763	2.993	$t(3)-t(4)$	3.392	4.041	2.319
rts1	RTS Index	14.432	6.716	3.179	$t(3)-t(4)$	4.982	4.547	2.411
rts2		29.391	6.508	3.087	$t(3)-t(5)$	4.177	5.333	2.789

Note: $var1$ is closing price, $var2$ is opening price. Period 1: April 01, 2019 – February 23, 2022 (N = 732), period 2: April 01, 2019 – February 20, 2020 (N = 200).

Table 3.

Values of kurtoses (2) and measures of heavy-tailedness (5) of index returns (logarithmic differences), *sectoral* indices

MOEX Index		Period 1				Period 2
		$K1$	$K1_{01}$	$K1_{05}$	Estimated $t(v)$ for residuals	$K2$
gaz1	Oil and gas	14.669	5.602	3.007	$t(3)-t(6)$	3.233
gaz2		27.428	6.221	3.388	$t(3)-t(5)$	3.529
chem1	Chemicals	6.772	5.785	3.528	$t(3)-t(6)$	5.527
chem2		10.558	6.034	3.270	$t(3)-t(5)$	7.789
electro1	Electric Utilities	21.194	6.604	3.107	$t(3)-t(4)$	3.519
electro2		36.633	6.775	2.937	$t(3)-t(4)$	4.044
telecom1	Telecoms	13.667	6.373	3.109	$t(3)-t(5)$	7.295
telecom2		51.869	7.142	3.461	$t(3)-t(4)$	5.977

MOEX Index		Period 1				Period 2
		$K1$	$K1_{01}$	$K1_{05}$	Estimated $t(v)$ for residuals	$K2$
metal1	Metals & Mining	12.399	5.621	2.842	$t(3)–t(6)$	3.295
metal2		25.064	5.975	3.049	$t(3)–t(5)$	3.472
finan1	Financials	13.063	5.973	3.309	$t(3)–t(5)$	4.097
finan2		37.475	6.557	3.327	$t(3)–t(5)$	4.421
potreb1	Consumer goods and Services	12.399	5.621	2.842	$t(3)–t(6)$	3.295
potreb2		25.064	5.975	3.049	$t(3)–t(5)$	3.472
trans1	Transport	16.832	7.081	3.457	$t(3)–t(4)$	5.569
trans2		18.288	7.712	3.368	$t(3)$	7.355

Note: *var*1 is closing price, *var*2 is opening price. Period 1: 01.04.2019–23.02.2022 (N = 732), period 2: 01.04.2019–20.02.2020 (N = 200).

sures of heavy-tailedness did not give a clear answer, but they allowed us to narrow down the number of models that need to be further evaluated.

5. Results

In this section, we compare the GARCH(1,1) specifications for the MOEX indices assuming different types of distribution in the residuals: normal and *t*-distribution with degrees of freedom from 3 to 9. The results of estimation and comparison of models for the indicator gaz1 are given as an example in *Table 3*. The likelihood ratio test [27, p.171] and the comparison of information criteria by Akaike and Schwartz were used to compare models. The results showed the same result, so only the value of the maximum of the likelihood function is given below in the text. Estimates of the parameters γ and β of the GARCH(1,1)-model in the form (3) are given in *Table 4*. LLF values are the value of the maximum likelihood function for the current model. Note that the full log-likelihood function with the inclusion of terms without optimization parameters is calculated in Stata. The form of the log-likelihood function under the assumption of normal and *t*-distribution is given in [28]. All model coefficients were statistically significant at the 1% significance level.

The results show (*Table 4*) that for various specifications, the coefficient $\beta \approx 0.84–0.89$, which indicates the persistence of volatility over time, $(\gamma + \beta)$ exceeds 0.9, which indicates the presence of a pronounced GARCH effect. The coefficients γ and β for various specifications behave quite steadily. $(\gamma + \beta) > 1$ is for the case *t*(3), which violates the condition of positivity of the conditional variance of the model. This model is also not adequate in terms of modeling heavy tails, since the kurtosis of the *t*-distribution is defined and greater than 3 for the degrees of freedom $v > 4$. The model with *t*(5)-distribution in the residuals is the best model in terms of the minimum values of information criteria, for which AIC = –4300.17 and BIC = –4281.79. For this model, there is also a maximum of LLF = 2154.09. GARCH(1,1) assuming a *t*(5) distribution in residuals would be the most preferred model for predicting vola-

Table 4.

Estimates of parameters of the GARCH(1,1) model for the variable gaz1 assuming different degrees of freedom of the *t*-distribution for residuals

Distribution of residuals	γ	β	$\gamma+\beta$	LLF
N	0.155	0.836	0.991	2130.876
$t(3)$	0.136	0.892	1.028	2149.819
$t(4)$	0.110	0.884	0.993	2153.393
$t(5)$	**0.104**	**0.879**	**0.982**	**2154.087**
$t(6)$	0.102	0.875	0.978	2153.702
$t(7)$	0.103	0.871	0.974	2151.963
$t(8)$	0.103	0.871	0.974	2151.963
$t(9)$	0.104	0.869	0.973	2150.997

tility. Note that this model is in line with the assumptions about model according the measures of heavy-tailedness: $t(3)$–$t(6)$ (*Table 3*).

Note that the indicator gaz1 has a kurtosis $K1 = 14.67$, which indicates outliers in the data and does not allow using the assumption of normality of the residuals in the GARCH(1,1)-model. GARCH(1,1) with normality in residuals has the highest AIC, BIC and the lowest LLF. Measures of heavy-tailedness $K1_{01} = 5.60$ and $K1_{05} = 3.01$ fall within the variation intervals for the measures of heavy-tailedness for distributions $t(3)$–$t(6)$ (*Table 1*), which in this case coincided with the results of estimating the GARCH(1,1)-model by the enumeration method.

GARCH(1,1)-models were evaluated in the same way for all other indicators. The best models with maximum LLF are given in *Table 5*. Comparison of various specifications of GARCH(1,1) models for each indicator is given in the Appendix.

Models with $t(5)$-distribution assumptions in residuals are the most common and best models as shown by the analysis of GARCH(1,1)-model parameter estimates for MOEX Indices. Such specifications of the model amounted to 60%, while the kurtosis of the logarithmic returns of indicators varied from 6 to 51 (*Tables 2, 3*). The ability to evaluate GARCH(1,1)-models under the assumption of a $t(v)$ distribution in residuals is available in Stata16 with $v \to \infty$ (for example, you can choose $v = 1000$). As the analysis of empirical data shows, the degrees of freedom v of t-distributions for the considered indicators vary from 4 to 7 (*Table 5*), and in terms of measures of heavy-tailedness at $v = 10$, the t-distribution approaches normal. Degrees of freedom assumptions based on measures of heavy-tailedness agreed with the results of empirical analysis by model enumeration in 68% of cases. A few indexes, for example trans2, became an exception. Models with an assumption of a $t(3)$-distribution in the residuals, for which there is no theoretical kurtosis, were not among the best. You can also notice that there is not a single model with the assumption of normal residuals for period 1. Such models had, as a rule, the worst LLF characteristics (application).

Table 5.

Estimates of the parameters of the GARCH(1,1)-model for MOEX Indices assuming different degrees of freedom of the *t*-distribution of residuals

MOEX Index	Estimated distribution of residues according to measures of heavy-tailedness (5)	Distribution of residuals of the best model by LLF	γ	β	$\gamma+\beta$	LLF
Main Equity Indice						
blue1	$t(3)-t(5)$	$t(6)$	0.115	0.869	0.984	2233.683
blue2	$t(3)-t(5)$	$t(5)$	0.123	0.861	0.984	2219.847
imoex1	$t(3)-t(5)$	$t(6)$	0.116	0.871	0.987	2295.943
imoex2	$t(3)-t(4)$	$t(7)$	0.112	0.863	0.975	2294.071
rts1	$t(3)-t(4)$	$t(5)$	0.097	0.901	0.998	2052.176
rts2	$t(3)-t(5)$	$t(5)$	0.121	0.871	0.992	2046.424
Sectoral Indices						
gaz1	$t(3)-t(6)$	$t(5)$	0.104	0.879	0.982	2154.087
gaz2	$t(3)-t(5)$	$t(5)$	0.145	0.84	0.985	2109.548
chem1	$t(3)-t(6)$	$t(5)$	0.05	0.955	1.005	2345.464
chem2	$t(3)-t(5)$	$t(5)$	0.149	0.829	0.978	2313.152
electro1	$t(3)-t(4)$	$t(5)$	0.136	0.842	0.978	2350.902
electro2	$t(3)-t(4)$	$t(5)$	0.137	0.843	0.98	2350.902
telecom1	$t(3)-t(5)$	$t(4)$	0.119	0.864	0.983	2437.535
telecom2	$t(3)-t(4)$	$t(4)$	0.152	0.857	1.009	2418.431
metal1	$t(3)-t(6)$	$t(5)$	0.096	0.879	0.975	2275.479
metal2	$t(3)-t(5)$	$t(5)$	0.142	0.806	0.948	2262.375
finan1	$t(3)-t(5)$	$t(5)$	0.102	0.898	1.000	2167.337
finan2	$t(3)-t(5)$	$t(4)$	0.124	0.883	1.007	2129.232
potreb1	$t(3)-t(6)$	$t(4)$	0.104	0.881	0.985	2274.596
potreb2	$t(3)-t(5)$	$t(5)$	0.142	0.806	0.948	2262.375
trans1	$t(3)-t(4)$	$t(4)$	0.212	0.732	0.944	2251.718
trans2	$t(3)$	$t(6)$	0.301	0.668	0.969	2217.500

Note that in this work we did not consider alternative approaches to estimating the heavy-tailedness distribution, for example, other types of measures, and this is also of scientific interest for further research. The GED-distribution is another possible distribution in the specification of GARCH-models, which was not considered in this study. Often the distribution of financial indicators has asymmetry, which also needs to be taken into account when choosing the specification of GARCH models, but we have not considered it. As noted above, measures of heavy-tailedness based on interfractile ranges are also used in works which require a sample size of $N \geqslant 1000$ [17]. These measures may also be the subject of further research.

Conclusion

We considered generalized autoregressive models of conditional heteroscedasticity for 22 MOEX Indices (Main Equity Indices and Sectoral Indices) with different kurtosis values from 3 to 52 in order to study the heavy-tailedness of distributions and the influence of kurtosis on the choice of the distribution type assumption in the residuals of the model to explain the fat tails. As the analysis showed, kurtosis is only partly an "indicator" of fat tails: on its basis, it is difficult to make an assumption about the form of the distribution of residuals, since it is sensitive to outliers. So, for example, the kurtoses for chem1 and blue2 were 6.77 and 33.43, but for these indicators the best model turned out to be the same model specification – GARCH(1,1) with t(5)-distribution in the residuals. It was shown in the work that the considered measures of heavy-tailedness are sufficiently robust to outliers and allow us to *partially* justify the choice of the degree of freedom for the t-distribution when evaluating GARCH(1,1)-models. It should be noted that the use of the model comparison approach based on maximum likelihood estimates gives similar results in 68% of cases (*Table 5*). Perhaps the classical approach is more preferable in econometric practice for the analysis of financial time series on samples of size $N < 1000$. However, the analysis of measures of heavy-tailedness is of great practical importance for modeling time series with heavy tails and substantiating the choice of degrees of freedom of the t-distribution, since kurtosis is not a good quantitative measure of the "heaviness" of distribution tails. In the opinion of the authors, measures of heavy-tailedness and their properties can be useful to a wide range of researchers working with financial time series in order to obtain more accurate profitability forecasts. This article is a small contribution to the further development of time series analysis tools. ■

Appendix.

Comparison of GARCH(1,1) models for MOEX Index by LLF

	MOEX Index										
Distribution of residuals	blue1	blue2	imoex1	imoex2	rts1	rts2	gaz2	chem1	chem2	electro1	electro2
N	2217.00	2189.29	2279.85	2277.39	2021.94	2017.52	2067.12	2321.80	2285.55	2321.49	2266.05
t(3)	2225.95	2213.30	2288.15	2284.26	2048.65	2041.62	2104.99	2342.08	2309.40	2347.75	2322.18
t(4))	2231.36	2218.40	2293.59	2290.61	2051.96	2045.63	2108.85	2345.10	2312.76	2350.74	2325.95
t(5)	2233.23	**2219.85**	2295.48	2293.09	**2052.18**	**2046.42**	**2109.55**	**2345.46**	**2313.15**	**2350.90**	**2326.57**
t(6)	**2233.68**	2219.83	**2295.94**	2293.95	2051.30	2046.02	2109.00	2344.89	2312.49	2350.07	2325.93
t(7)	2233.51	2219.18	2295.78	**2294.07**	2050.04	2045.15	2107.95	2343.97	2311.44	2348.89	2324.75

Distribution of residuals	MOEX Index										
	blue1	blue2	imoex1	imoex2	rts1	rts2	gaz2	chem1	chem2	electro1	electro2
$t(8)$	2233.06	2218.27	2295.33	2293.83	2048.67	2044.10	2106.71	2342.94	2310.28	2347.63	2323.37
$t(9)$	2232.49	2217.26	2294.77	2293.41	2047.32	2043.02	2105.41	2341.91	2309.12	2346.38	2321.91

Distribution of residuals	MOEX Index									
	telecom1	telecom2	metal1	metal2	finan1	finan2	potreb1	potreb2	trans1	trans2
N	2383.40	2355.57	2244.45	2216.95	2142.97	2077.17	2244.45	2259.31	2198.72	2192.53
$t(3)$	2435.90	2416.07	2270.33	2259.31	2166.45	2126.57	2270.33	2259.31	2250.95	2210.66
$t(4)$	**2437.54**	**2418.43**	2274.60	2262.29	2166.45	**2129.23**	**2274.60**	2262.29	**2251.72**	2215.79
$t(5)$	2436.19	2417.67	**2275.48**	**2262.38**	**2167.34**	2129.05	2275.48	**2262.38**	2249.93	2217.34
$t(6)$	2433.95	2415.87	2275.07	2261.43	2167.05	2127.84	2275.07	2261.43	2247.41	**2217.50**
$t(7)$	2431.52	2413.74	2274.14	2260.10	2166.30	2126.24	2274.14	2260.10	2244.78	2217.06
$t(8)$	2429.13	2411.56	2273.01	2258.66	2165.37	2124.53	2273.01	2258.66	2242.26	2216.37
$t(9)$	2426.88	2409.43	2271.83	2257.22	2164.41	2122.84	2271.83	2257.22	2239.91	2215.57

Note. For gaz1, a comparison of the models is given in *Table 4*.

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