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Prediction of distributions of unit prices for real estate properties on the basis of the characteristics of PSI-processes*

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Abstract

Real estate market price forecasting is always in the focus of interests of scientists-economists, market analysts, market participants (sellers and buyers), marketing services of building complex enterprises, analysts working for banks and insurance companies and investors. Under present day conditions, the price behavior of properties on real estate markets takes especially important meaning subject to the influence of such factors as changes in the structure of household incomes, changes in mortgage rates and their availability, dynamic changes in the macroeconomic and other external socio-economic and

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political type factors. However, unlike the financial and securities markets, the real estate market is always characterized by a delayed reaction to external perturbations, often up to half a year, which allows us to hope for an appropriate construction of forecasts, at least in time for the delayed reaction. Traditional autoregressive forecasting methods are characterized by rapidly increasing forecast variance, because they assume a factor of stochastic volatility. This paper proposes a model and method of forecast construction based on stochastic processes of the "Poisson random index" having a short time for reaching a stationary stable variance. The model is based on the "principle of replacements" of current prices with new ones. We analyze in detail an example of the application of the "principle of replacements" for construction of price forecasts on secondary residential real estate in St. Petersburg which is based on data of four-year observations of offer prices.

Keywords: real estate price forecast, log-normal price distribution, pseudo-Poisson process, Poisson random index process, Ornstein–Uhlenbeck process

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Introduction

The real estate market is an important sector of the economy of any country. Capital construction projects entering the turnover of the commodity real estate market create chains of economic relations in the construction industry, in the sectors of construction materials production, and in the extractive industry. The sphere of turnover of primary and secondary real estate properties is to a large extent the area of business interests of the banking and insurance sectors of the economy. The aggregate of real estate properties owned and in commercial turnover is the taxable base for property tax and, consequently, a basis for replenishment of the country's budget. For these reasons, forecasting prices in the real estate market has always been and remains an urgent task for all market participants. Materials devoted to this subject regularly appear in periodicals. The tasks related to the construction of forecasts are constantly the subject of research by scientists and researchers. Among the works of a general methodological nature, it is worth mentioning the well-known translated book [1], a series of works by the authors of publications [2–5]. As examples of relatively recent publications of domestic researchers, we can name works [6–16] and foreign works – [17–22]. Real estate markets, as rightly noted in [5], have characteristic regional features. The works of domestic authors [3, 6, 10, 11, 13–16, 23, 24] and foreign authors – [25–28] are devoted to price forecasting in regional domestic markets. In most cases, traditional forecasting methods are considered: fundamental and technical analysis [11], factor models [14], regression models, autoregressive and moving average models [2–5, 15, 16, 18, 27]. At the same time, "standard econometric methods are unsuitable for forecasting real estate market trends in modern conditions" and "methods developed in countries with developed market economies are unsuitable for forecasting in countries with transition economies" [5]. Recently, there are works in which machine learning methods are applied for the purposes of real estate price forecasting [24], including neural network modeling methods [8, 29]. The main disadvantage of traditional methods based on autoregression is a rapid increase in the forecast variance which makes the forecast result uninformative after 2–3 steps. This is due to the introduction of a random scale of volatility, which is reflected in the fact that a mixture of distributions arises – and the variance of the mixture is always greater than or equal to the mixture of variances. At the same time, numerous empirical observations suggest that the variance of unit prices of relatively similar real estate properties changes little over time, even in the presence of strong upward or downward trends. The exception may be relatively short periods of exposure to strong external perturbing factors leading to a noticeable change in the trend. In this regard, a forecast with a relatively stable variance would be preferable.

In this paper, we propose a mathematical model of the price change process in the real estate market under simple and seemingly natural assumptions. Such assumptions are the following statements:

1) the announced selling price for some property remains unchanged during some (random) time interval;
2) at any point in time, the property may be withdrawn from sale or sold;
3) at any moment of time, a new property may appear in the listing of properties for sale and replace the property withdrawn from sale or sold;
3) the prices of the properties in the listing at each fixed point in time are independent, or at least conditionally independent subject to some external factor;
4) the total set of properties for sale at a fixed point in time forms an observable sample of unit prices, which is subject to study and statistical processing.

The theoretical basis of the model proposed below is based on three provisions:

- the principle of log-normal distribution of unit prices of relatively homogeneous real estate properties;
- characteristics of the distribution and covariance function of the Poisson random index process (hereinafter referred to as PSI-process);
- the central limit theorem for PSI-processes – convergence of their normalized sums to the stochastic Ornstein–Uhlenbeck process (stationary, Gaussian, Markov process).

In [30, 31] the justification of the convergence of unit prices formed by successive comparisons to a log-normal distribution is given. Apparently, the first work in which noted the adherence of unit rental rates to a log-normal distribution was the work of British statisticians [32]. This adherence is also noted in the work of Japanese scientists [26]. In [33] the characteristics of distributions and correlation functions of PSI-processes are investigated, and the convergence of normalized sums of independent PSI-processes to Ornstein–Uhlenbeck type processes is proved there. The theoretical foundations of the model proposed in this paper are presented in [30, 31, 33, 34].

1. Definition and basic characteristics of the PSI-process

Let $\xi_0, \xi_1, \xi_2, \ldots$ be a random sequence, which we will call the forming or slave sequence; $\Pi(t) = \Pi_\lambda(t)$, $t \geqslant 0$ is a Poisson process independent of it with intensity $\lambda > 0$, which we will call the master. We define a Poisson subordinator1 for the sequence $\xi_0, \xi_1, \xi_2, \ldots$ as follows $\psi(t) = \psi_\lambda(t) := \xi_{\Pi(t)}$. The resulting process $\psi(t)$ with continuous time $t \geqslant 0$ – we will call the Poisson Stochastic Index process or PSI-process. Note that PSI-processes are a natural generalization of the pseudo-Poisson processes introduced and discussed in detail in Chapter X of the second volume of Feller's classic work [35].

The PSI-process represents successive replacements of the members of the forming sequence, occurring at the moments of jumps of the Poisson process. Time intervals $\{\tau_{j+1}\}, j = 0, 1, 2, \ldots$ between consecutive jumps of the leading Poisson process are called spacings. It is known that spacings are independent identically distributed random variables with a common exponential (or, what is the same, exponential) distribution having intensity $\lambda > 0$. At time zero, ξ_0, which "holds" its value during the first spacing, is played out, at the time of the first jump of the Poisson process it is replaced by ξ_1, and so on.... During the j-th spacing, the played random variable $\{\xi_{j-1}\}$ does not change its value until (inclusive

1 Subordination in theories of stochastic processes is called the generally accepted random replacement of time.

of) the moment of the j-th jump of the Poisson process θ_j. At the moment of the spacing change τ_{j+1} to τ_{j+2}, the value of the random variable ξ_j is replaced by ξ_{j+1} (*Fig. 1*).

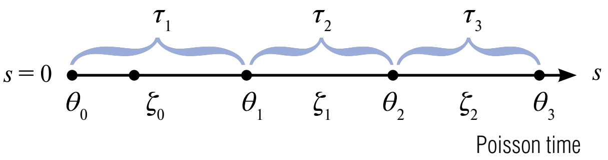

Fig. 1. Schematic representation of the random spacing lengths $\tau_0, \tau_1, \tau_2, \ldots$, the moments of the Poisson process jumps $\theta_0, \theta_1, \theta_2, \ldots$ and the substituted random variables $\xi_0, \xi_1, \xi_2, \ldots$.

Note that the PSI-process has the following representation as a weighted sum of elements of the random sequence $\xi_0, \xi_1, \xi_2, \ldots$, where the weights are Poisson indicators

$$\psi_\lambda(t) = \sum_{j=0}^{\infty} \xi_j \cdot \mathbf{1}\{\Pi(t) = j\},$$

hereafter **1** is the indicator function.

If $\xi_0, \xi_1, \xi_2, \ldots$ represent a stationary sequence, then the PSI-process is stationary. In particular, this will be fulfilled when $\xi_0, \xi_1, \xi_2, \ldots$ are independent identically distributed quantities. In the latter case, if $E\xi_0 = a$, then the conditional mathematical expectation of the PSI-process is

$$E\left(\psi_\lambda\left(t+s\right) \mid \psi_\lambda(s) = z\right) = ze^{-\lambda \cdot s} + a(1 - e^{-\lambda \cdot s})$$

for any arbitrary but fixed z, for any non-negative t, s. And, in particular, when $a = 0$

$$E\left(\psi_\lambda\left(t+s\right) \mid \psi_\lambda(s) = z\right) = ze^{-\lambda \cdot s}.$$

It is important to note that if the sequence $\xi_0, \xi_1, \xi_2, \ldots$ is Markovian, then the corresponding PSI-process also has the Markov characteristic[2].

2. The covariance function of the process $\psi(t)$

In [33] the following result is obtained. Let $\xi_0, \xi_1, \xi_2, \ldots$ are independent identically distributed random variables, $E(\xi_0) = 0$, $D(\xi_0) = 1$, then the covariance function of the process $\psi(t)$, $t \geqslant 0$ decreases exponentially and has the form

$$cov(\psi(t), \psi(0)) = \exp(-\lambda t).$$

Consider independent copies of a single PSI-process $\psi_\lambda(t)$: $\psi_1(t), \psi_2(t), \ldots$ It follows from the exponential form of the covariance function that, by virtue of the central limit theorem (CLT) for vectors, a random process composed of normalized sums of PSI-processes of the form

$$Z_N(t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N} \psi_{(j)}(t),$$

converges in the sense of weak convergence of finite-dimensional distributions[3], when $N \to \infty$, to the Ornstein–Uhlenbeck process, a stationary, Gaussian, Markov process, and given in a "standardized" form. The latter means that at arbitrary moments of time $(t_1, \ldots, t_d)$, the vector $Z(t_1), \ldots, Z(t_d)$ has a joint normal distribution with zero mean and with covariance $\exp(-\lambda|t_i - t_j|)$, where t_i, t_j are running all elements of the set$(t_1, \ldots, t_d)$. Thus, the distribution of the "standardized" Ornstein–Uhlenbeck process is characterized by Gaussianity (joint normality of finite-dimensional distributions), zero mean (zero "theoretical" trend), and covariance of the form $\exp(-\lambda t)$. The coefficient $\lambda > 0$ is called the "speed" of the Ornstein–Uhlenbeck process, and $1/\lambda$ is called the "viscosity". Here we see that the rate of the limiting Ornstein–Uhlenbeck process is exactly the intensity of the "leading" Poisson process. The conditional expectation of the Ornstein–Uhlenbeck process coincides with the conditional expectation of the PSI-process

$$E\left(U(t+s) \mid U(t) = z\right) = ze^{-\lambda \cdot s},$$

[2] The Markov characteristic is when the future, with a fixed past and present, does not depend on the past (that is, it depends on the past only through the present).

[3] Moreover, there is convergence of $Z_N(t)$ in the Skorokhod functional space, see [34].

and the conditional variance is independent of the z condition and is equal to

$$D\big(U(t+s)\,|\,U(t)=z\big)=1-e^{-2\lambda\cdot s}.$$

Moreover, the conditional distribution of the process U given z is normal at any non-negative s. From the standard Ornstein–Uhlenbeck process by shifting a and scaling $b > 0$, we obtain an Ornstein–Uhlenbeck process OU which has a stationary distribution normal with parameters a and b^2, covariance $b^2\exp(-\lambda t)$. For such an Ornstein–Uhlenbeck process, of course, stationarity and the Markov property are preserved, and the conditional expectation and conditional variance are respectively equal to

$$E\big(OU(t+s)\,|\,OU(s)=y\big)=a+(y-a)e^{-\lambda\cdot s},$$
$$D\big(OU(t+s)\,|\,OU(t)=y\big)=b^2(1-e^{-2\lambda\cdot s}). \quad (1)$$

From where, in particular, we can see that the conditional variance of OU is independent of the condition.

Applied to real estate prices, we interpret the model under consideration as follows. We represent the price listing as a table, where each row is a time slice of current offer prices (N is the volume of the slice, changes from slice to slice), and each column is the price of the property in dynamics, possibly with correction for trend (n is the number of slices). We observe price slices with some periodicity determined by the next issue of the price log (usually once a week). By the next issue of the log, each object may "go away" (or dramatically "go away" its price), or it may stay with the same price. A property, or its price, can be replaced by a newly arrived property (price). Moreover, the remaining prices from the previous issue of the journal are significantly more than the newly arrived ones in the subsequent issue. Each PSI-process is represented by a column in a table. The values are observed in time slices, according to the dates of the next issue of the journal. All PSI-processes are assumed to be independent and identically distributed (i.e., they are independent copies of a single PSI-process – in our context, price, or the logarithm of price). The distribution of each PSI-process is understood as the distribution of a piecewise constant function with continuous time, continuous on the right, having finite limits on the left (such functions are called Right Continuous Left Limits, RCLL). The Poisson process acts as a point process determining the moments of replacements due to its characteristic of "no aftereffect": no matter how much time has passed since the previous jump, the next jump will occur after an exponential time.

In our approach, we make the following ***approximation of the logarithms of prices*** in the "table". Let $V(t)$ be the price of 1 square meter of the selected property type at time t. In each slice at time t we perform logarithmization and observe independent realizations of the process $\ln(V(t))$ with added trend and scale factor $\sigma > 0$. Thus, at each time slice t we have (our) simple sample of size equal to the number of prices in the given time slice.

The method of constructing forecasts of price distributions and their numerical characteristics, proposed in this paper, is based on specific aspects of PSI-processes and limits of their normalized sums: Ornstein–Uhlenbeck processes. The main general properties here (for both the PSI-process and the Ornstein–Uhlenbeck process) are Markovianness, the kind of conditional mathematical expectations. We also use the type of conditional variance and the Gaussian property of the conditional distribution of the Ornstein–Uhlenbeck process.

Constant monitoring of prices on the real estate market, with a predetermined periodicity, allows us to use the characteristics of the Ornstein–Uhlenbeck processes to forecast future price distributions and their numerical characteristics, such as mean, median, modal (market value) values, standard deviation, corridors of acceptable values, and so on. The peculiarity of the real estate market is that price changes on the market can be observed, as a rule, not more than once a week, as most printed advertising publications and Internet resources are updated with the same frequency. At the same time, the real estate market has a slow reaction to external sources of disturbances of macroeconomic nature, as the search for an object, reaching agreements, registration of the transaction, entry into the ownership rights require considerable time. In this regard, it is quite reasonable to monitor prices with a periodicity of once a month, because during such a period of time price changes become noticeable.

3. Model

Let us consider the random process $V(t)$ – the dynamics of the price of 1 sq. m. of real estate in time and the associated process of the price logarithm $Y(t) = \ln(V(t))$. We take the average of each slice as the basic estimate of $Y(t)$ and denote it by the same letter (this also applies to $\hat{Y}(t)$, introduced below). Suppose the process $Y(t)$ has a linear trend $E(Y(t)) = \alpha \cdot t + \beta$ and a time-constant standard deviation $\sigma(t) \equiv \sigma > 0$. We will consider the centered and normalized process

$$\hat{Y}(t) = \frac{Y(t) - \alpha \cdot t - \beta}{\sigma}$$

as a PSI-process. Note that under the formulated assumptions $\hat{Y}(t)$ has a mathematical expectation identically equal to zero and a variance equal to one. Under the condition that the distribution (as a random process) of $\hat{Y}(t)$ coincides with the distribution of the PSI-process, or Ornstein–Uhlenbeck process, the covariance function is $cov\left(\hat{Y}(t)(s+t), \hat{Y}(t)\right) = \exp(-\lambda s)$ and does not depend on t. Since observations of prices in the real estate market are only available in discrete time, we speak of the values and observations of the process $V(t)$ at discrete points in time $V(j)$, $j = \overline{0, n-1}$. In [33, 34], weak convergence of the distribution of prices formed by successive comparisons to a log-normal distribution was proved. Thus, we have reasons to consider the sequence of price logarithms $Y(j) = \ln(V(j))$, $j = \overline{0, n-1}$ as a sequence of normally distributed random variables, whence it follows that all members of the corresponding sequence $\hat{Y}(t)$ are standardly normal. In this context, the physical meaning of the members of the sequence $(\xi_j), j = \overline{0, \infty}$ (from the definition of the PSI-process) can be the centered, standard deviation normalized logarithms of the price of 1 sq. m. of some property maintaining its value during the time interval (τ_i), $i = \overline{1, \infty}$, $(i = j + 1)$, i.e., a sequence of the form $\hat{Y}(j)$, $j = \overline{0, n-1}$. Under the assumption that the mean values of the price logarithms follow the Ornstein–Uhlenbeck process, for any pair of mean values of the logarithms of the random variables $V(t)$, $V(t+s)$, the hypothesis of a multivariate joint normal distribution for $\ln(V(t))$ can be considered. Note that quantiles (in particular, median) and mode can be considered as mean values here. If this hypothesis is confirmed for any predetermined $V(0) = v(0)$, forecast estimates can be obtained for the modal, median and mean values of the random variable $V(s)$ using the formulas of conditional mode, median and/or conditional mathematical expectation (see, for example, [36, 37]). Here, time 0 corresponds to the moment of the last observed price distribution, s is the time for which the forecast is given, its counting starts from the moment of the last observed distribution.

To verify the proposed interpretation, the following statements should be statistically proved and confirmed.

1) Study the distribution of spacings (τ_i), where $i = \overline{1, n}$ – number of periods of continuous price presence in the flow (the purpose is to obtain statistical confirmation of the hypothesis about their exponential distribution, to estimate the exponential distribution parameter).
2) Study price distributions $V(j)$, $j = \overline{0, n-1}$ in each slice (the purpose is to be convinced of the log-normal form of the price distribution);
3) Obtain confirmation of the independence of the copies of the random sequence $\hat{Y}(j)$, $j = \overline{0, n-1}$ for which purpose we study the behavior of the accumulated variances in each slice. We will check for uncorrelatedness, which together with Gaussianity will give independence.
4) Verify the characteristic of joint normality (Gaussianity) of the mean (for each slice) values of $\hat{Y}(t)$, which, together with the condition on the exponential form of the covariance function, confirms the Markov property of the random sequence composed of the mean (for each slice) of $\hat{Y}(j)$, $j = \overline{0, n-1}$. This follows from the fact that a stationary Gaussian random process with correlation decreasing exponentially is an Ornstein–Uhlenbeck process, and hence it has the Markov property. Moreover, the Gaussianity of the mean will confirm the Markov property of the median and mode as a function of $j = \overline{0, n-1}$.

Note that the Gaussianity of the mean for each slice is equal to the Gaussianity of the sums for each slice normalized by $\sqrt{N}$ for our range of values of N:

from 254 to 729. Also note that the Gaussianity of the averages may follow from the fact that the Ornstein–Uhlenbeck process in discrete time is a 1st order autoregression, and for uncorrelated noise it is sufficient to show conformity to the normal law.

4. Methodology of forecast construction[4]

1) Construct a series from the ***mean values*** of all time slices, identify the trend, subtract it from the series, obtain an estimate of the centered and standardized series $\hat{Y}(j),\ j=\overline{0, n-1}$.
2) For the estimated series $\hat{Y}(j),\ j=\overline{0, n-1}$, construct the partial autocovariance function (PACF), obtain confirmation of the hypothesis of conditional uncorrelation of $\hat{Y}(j)$ with $\hat{Y}(k),\ |k-j|>0$.
3) For the estimated series $\hat{Y}(j),\ j=\overline{0, n-1}$, construct the covariance function (autocovariance function, ACF), obtain confirmation of the hypothesis about the exponential form of the covariance function, estimate the exponent's degree parameter, and compare it with the exponential distribution parameter of the observed spacings.
4) Using formulas of the form (2)–(5) written below, construct a forecast.

Let us introduce the notations:

$E(\ln(V(0)))=\mu(0),\ E(\ln(V(s)))=\mu(s),$

$\sigma(\ln(V(0)))=\sigma(0),\ \sigma(\ln(V(s)))=\sigma(s),$

$\rho(\ln(V(0)), \ln(V(s)))=e^{-\lambda\cdot s}.$

Under the assumption that there is a linear trend of the form $\mu(s)=\alpha\cdot s+\beta$ for the mean logarithms of prices, the standard deviation is constant $\sigma(s)=\sigma(0)=\sigma$, we obtain for any predetermined value $V(0)=v(0)$ the forecast estimates:

$$Mode(V(s)|V(0)=v(0)) = \exp(\alpha\cdot s+\beta+e^{-\lambda\cdot s}(\ln(v(0))-\mu(0)-\sigma^2(1-e^{-2\lambda\cdot s})), \quad (2)$$

$$Median(V(s)|V(0)=v(0)) = \exp(\alpha\cdot s+\beta+e^{-\lambda\cdot s}(\ln(v(0))-\mu(0))), \quad (3)$$

$$E(V(s)|V(0)=v(0)) = \exp(\alpha\cdot s+\beta+e^{-\lambda\cdot s}(\ln(v(0))-\mu(0))+\frac{1}{2}\sigma^2(1-e^{-2\lambda\cdot s})). \quad (4)$$

We also give a formula describing the bounds $V_{L,R}(s)$ of the error corridor (e.g., within one standard deviation):

$$V_{L,R}(s)=\exp(\alpha\cdot s+\beta+e^{-\lambda\cdot s}\left(\ln(v(0))-\mu(0)\right)\pm\sigma\cdot\sqrt{1-e^{-2\lambda\cdot s}}), \quad (5)$$

where R, L are indices denoting, respectively, the right and left boundaries of the error corridor.

Formulas (2)–(5) are a direct consequence of the important property that the conditional distribution of the *OU* process is normal with parameters given by (1).

5. A practical example of the application of the method

The data on apartment sales in St. Petersburg published in issues 1483 through 1686 of the St. Petersburg Real Estate Bulletin covering the period from September 2011 through October 2015 are selected for the example. The St. Petersburg Real Estate Bulletin was published in print weekly through the end of 2019. The issues were selected on a one-issue-per-month basis, totaling 50 issues. Obviously, the full selection of a monthly issue contains mixed information about properties of different categories, and our example requires prices for properties of approximately the same categories. For this purpose, properties located in the Admiralteysky District of St. Petersburg were selected from citywide information. The Admiralteysky District is characterized by extensive "old stock" development, with relatively few premium properties in the central part of the city or old stock overlooking the great Neva

[4] Below we continue to use the already accepted designations, but in relation to the processed data.

River. Premium properties are also excluded from the sample. We are interested in properties that were in the journal for at least one period, so the stream consists of 48 samples (hereafter we will call them time slices), which are empirical samples of realizations of random variables $Y(j) = \ln(V(j)), j = \overline{0, 47}$ and cover a period of 4 years. The standard deviations of all flow slices have a mean value of $\sigma = 0.22$ with insignificant changes over time (the standard deviation from its mean (0.22) is of order 0.01). Given the insignificant variations in the standard deviations, we will assume the standard deviation in the flow to be constant, equal to $\sigma = 0.22$ and make the prediction under this condition.

6. Spacing distribution

For each price value, the number of periods during which this price was maintained from slice to slice without a break was calculated. The resulting number is the length of the spacing τ corresponding to the given price, expressed in the total number of periods (one period – one month), taking integer values from 1 to 48. We then plot the empirical distribution of spacings by length (1, 2, 3, 4, ...), as well as their relative frequency to the total number of all spacings. The accumulated relative frequencies give the observed values of the empirical distribution function for spacing length. Our assumption is that the theoretical distribution of spacings obeys an exponential distribution law[5], that is, $F(t) = P(l \leqslant t) = 1 - e^{-\lambda \cdot t}$, where l is the random (theoretical) spacing length. We consider an additional probability distribution function $P(l > t) = 1 - F(t) = e^{-\lambda \cdot t}$. Obviously, $\ln(1 - F(t)) = -\lambda \cdot t$, i.e., the logarithm of the additional function depends linearly on t. Thus $\ln(1 - F(0)) = 0$. This property is used to fit the parameter λ of the exponential spacing distribution. *Figure 2* shows the observed values of the $1 - F(t)$ function at values $t = 0, 1, 2, 3, 4, 5, 6, 7, 8$ and their approximation by an exponent of the form $e^{-\lambda \cdot t}$.

The estimation of the parameter λ is obtained by the library function lm of the statistical package *R,* the value of $\lambda = 0.6510$.

7. Price distributions

Figure 3 shows the empirical distributions of prices of 1 sq. m. in the first six slices $V(j), j = \overline{0, 5}$.

Figure 4 shows the p-value values of Kolmogorov–Smirnov tests for the correspondence of empirical distributions in 48 slices to the theoretical log-normal distribution, with the selected parameters.

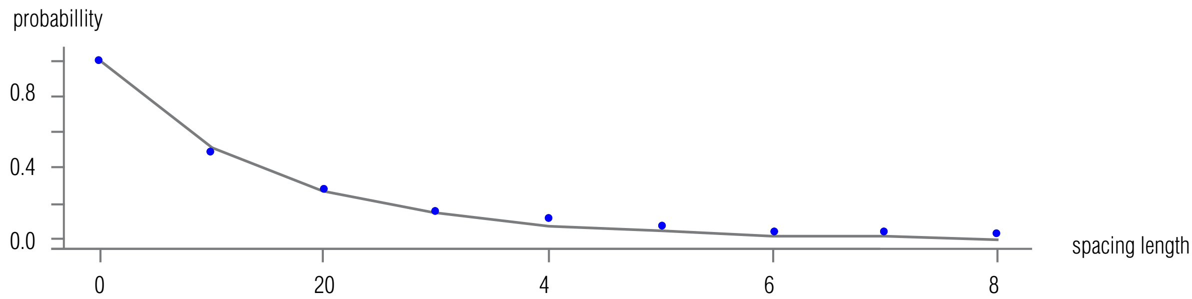

Fig. 2. Observed values of the additional function $1 - F(t)$ at values $t = 0, 1, 2, 3, 4, 5, 6, 7, 8$ (points) and their approximation by an exponent of the form $e^{-\lambda \cdot t}$.

[5] Since we consider slices at discrete time points, we observe geometrically distributed empirical spacings, which are the projection of continuous exponential spacings at discrete time points 0,1,2,3,... If the random variable is, $\xi \in \exp(\lambda)$, then $[\xi] \in Geom(p)$ (where square brackets denote the integer part of the number) and in our case, $p = 1 - e^{-\lambda}$.

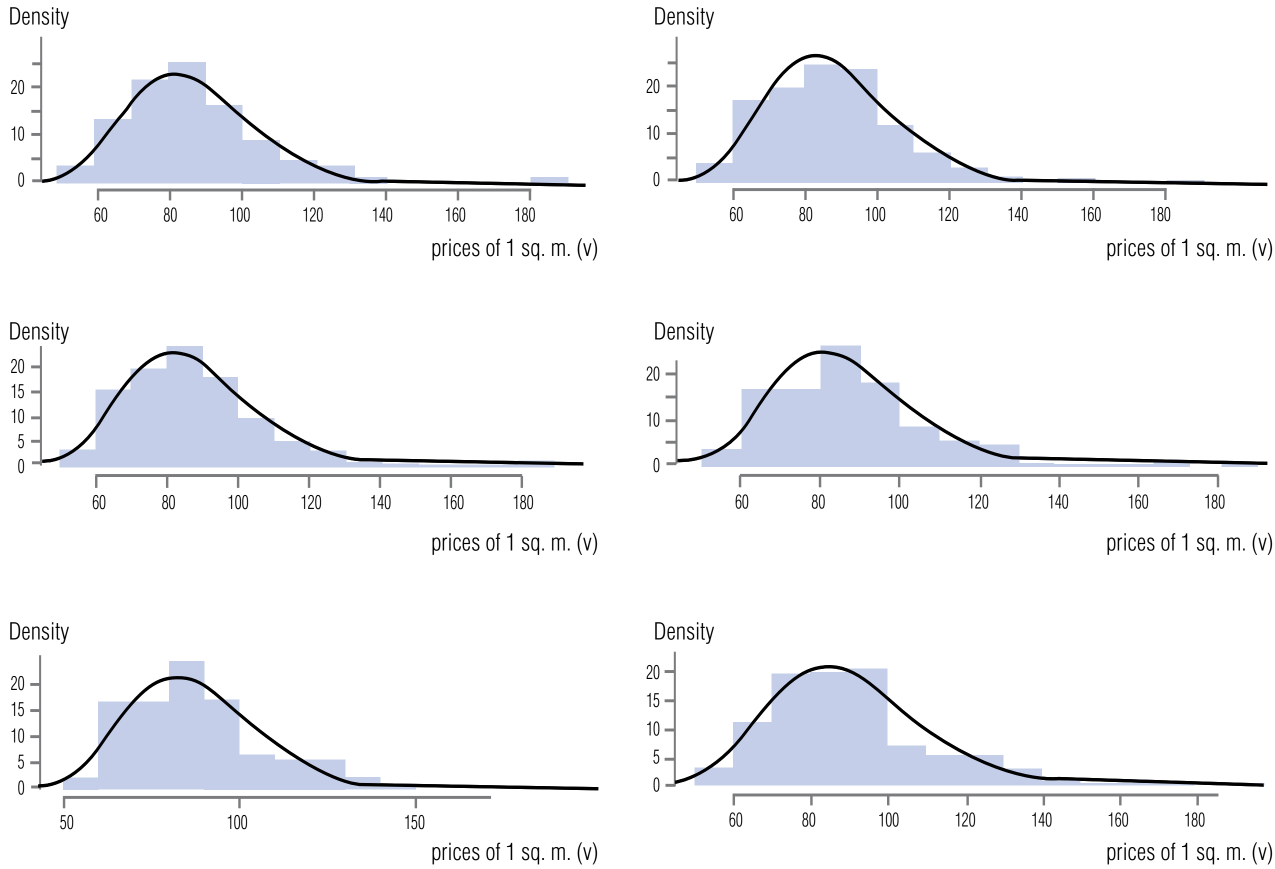

Fig. 3. Empirical distributions of prices of 1 sq. m. in the first six slices $V(j), j=\overline{1,6}$,
the line is the density of the model log-normal distribution.

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 [1] 0.32078110 0.20548610 0.40557570 0.32223009 0.26319875 0.22332022
 [7] 0.16753867 0.12097452 0.07531444 0.08037770 0.15068216 0.13773904
[13] 0.15556880 0.76906843 0.26749012 0.44678095 0.30453727 0.46525538
[19] 0.27115174 0.42116513 0.21523598 0.29461480 0.14404826 0.85821447
[25] 0.60624471 0.44881713 0.47551761 0.27010204 0.30031997 0.11477638
[31] 0.11761958 0.35463007 0.41267475 0.25858806 0.09053699 0.27756816
[37] 0.13531391 0.17893827 0.68449172 0.91269908 0.53167338 0.16490183
[43] 0.40555451 0.24649549 0.11753382 0.17232872 0.08156706 0.25339992
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*Fig. 4.*Screenshot of the R statistical package window with p-value values
of Kolmogorov–Smirnov tests for the correspondence of empirical distributions in 48 slices $V(j), j=\overline{0,47}$
of the theoretical log-normal distribution, with the selected parameters.

Thus, there is no reason to reject the hypothesis of log-normal distributions of prices in all 48 slices (or normal distributions of logarithms of prices).

8. Independence of copies of the random sequence

Let's consider the independence of copies of a random sequence $Y(j) = \ln(V(j)), j = \overline{0,47}$. We first note that the number of copies in each slice is different and varies from 254 to 729. It is known that the variance of the sum of two random variables is equal to the sum of the variance plus twice the covariance. We require evidence of uncorrelatedness of the records in the slices. To this end, we examine the behavior of the accumulated variance in each of the 48 slices $\ln(V(j))$, $j = \overline{0,47}$.

In the sample of each slice, the following steps are performed:

- a subsample is formed by randomly selecting a fixed number of elements from the sample (slice) (e.g. 20, the minimum sample size in the slice is 254, the maximum is 729), the variance is calculated;
- the next 20 elements are selected from the remaining elements of the sample, the variance is calculated, the result is added to the variance obtained at the previous step,
- the procedure continues until the sample is exhausted (the number of such steps is marked on the horizontal axis in *Fig. 5*).

Then the dependence of the variance accumulated in this way on the number of steps is considered. The linear character of the accumulated variance indicates the absence of a correlation term in the obtained sums. Since in our case the summarized quantities obey the normal distribution law, the non-correlation indicates independence.

Figure 5 shows the behavior of accumulated variance in the first slice, the behavior of accumulated variance in the first slice when the random selection procedure is repeated 1000 times, and the behavior of accumulated variance in all other slices.

9. Forecast construction

To construct the forecast, we should identify the linear trend in the data, estimate the PSI-process parameter λ, and check the joint normality of the mean values, which will provide confirmation of the Markov property of the observed process[6] . The standard deviation of the process is assumed above to be constant and equal to $\sigma = 0.22$.

Figure 6 shows a series of mean values of logarithms $Y(j) = \ln(V(j), j = \overline{0,47}$.

The linear trend is estimated using the library function lm of the statistical package *R*. We remove the linear trend, normalize the data by the standard deviation $\sigma = 0.22$, and average the data in each slice.

We obtain a series (*Fig. 7*) of average values of the process $\hat{Y}(j), j = \overline{0,47}$. The graph visually corresponds to the trajectory graph of the stationary process.

Figure 8 shows a plot of the autocorrelation function for a series of mean values of the process $\hat{Y}(j), j = \overline{0,47}$, with lags up to 7 and its approximation of the exponential of the form $e^{-\lambda \cdot s}$.

The estimation of the parameter $\lambda = 0.6502$, is obtained by applying the library function lm of the statistical package *R*.

Figure 9 shows the plot of the partial autocorrelation function for a series of mean values of the process $\hat{Y}(j), j = \overline{0,47}$, with lags up to 7.

[6] The Markov property is also indicated by the type of pacf in *Figure 9*, where there is exactly one significant peak per unit.

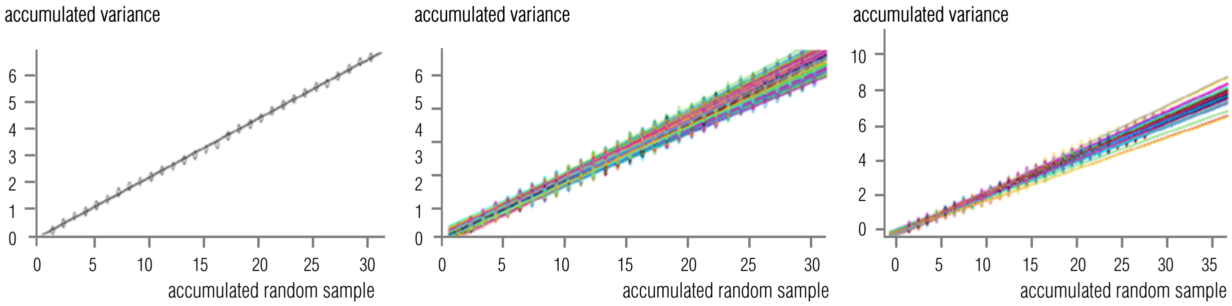

Fig 5. Left figure – accumulated variance of slice $\ln(V(1))$,
center figure – accumulated variance of slice $\ln(V(1))$
at 1000 repetitions of the random subsample selection procedure,
right figure – accumulated variance of all slice $\ln(V(j)), j = \overline{0, 47}$.

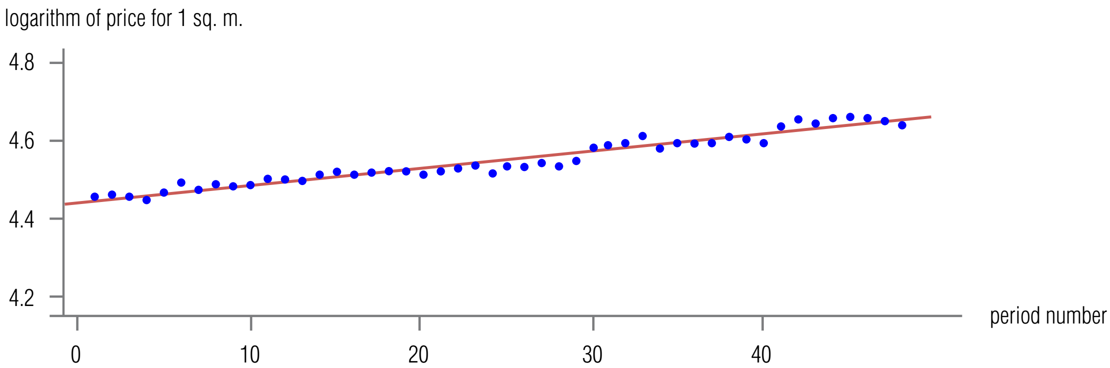

Fig. 6. Series of mean values for $Y(j) = \ln(V(j)),\ \ j = \overline{0, 47}$,
trend line equation $4.443 + 0.00432 \cdot t$, where t is the time in periods (1 period = 1 month)
from the beginning of observations (with j = 1, the value of t = 1/4).

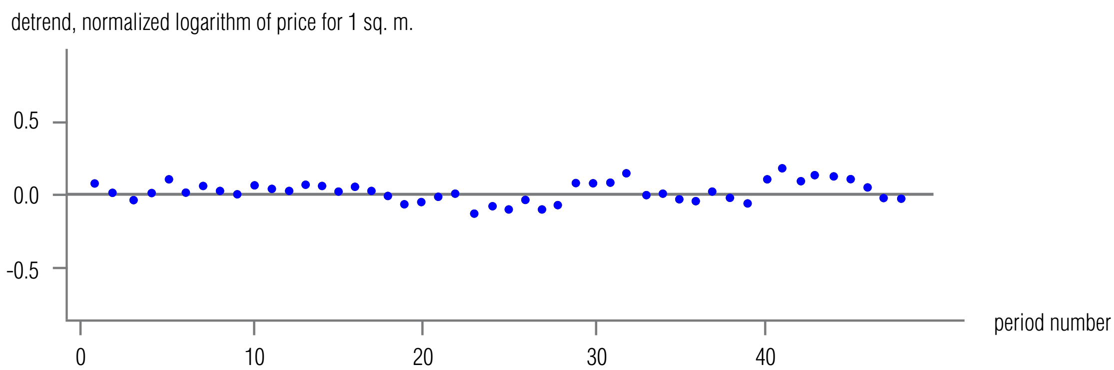

Fig. 7. A series of mean values of the process $\hat{Y}(j), j = \overline{0, 47}$.

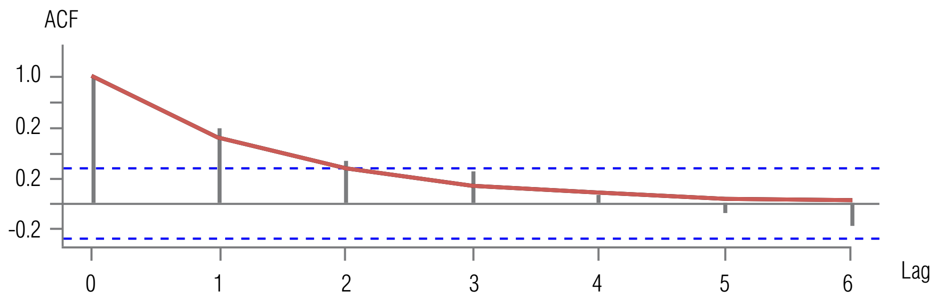

Fig. 8. Graph of the autocorrelation function for a series of mean values of the process $\hat{Y}(j), j=\overline{0,47}$, with lags up to 7 and its approximation by an exponent of the form $e^{-0,6502\cdot s}$.

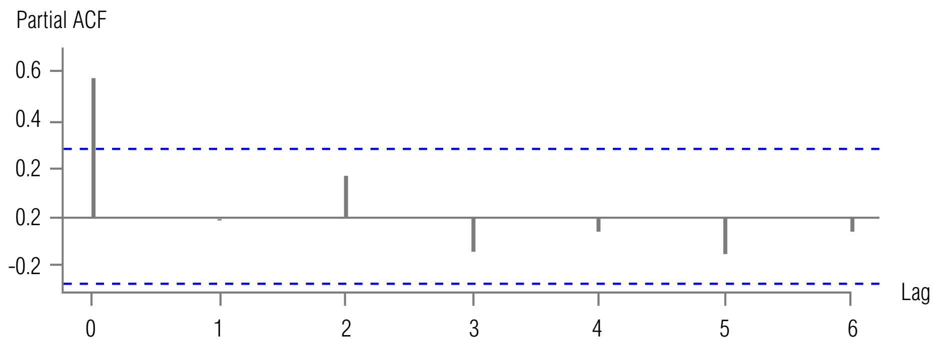

Fig. 9. Plot of the partial autocorrelation function for a series of process mean values $\hat{Y}(j), j=\overline{0,47}$, with lags up to 7.

10. Checking the joint normality of the mean values of $\hat{Y}(j), j=\overline{0,47}$

In the case of continuously substituted objects and prices, different sample sizes in each slice, checking joint normality by library tests (such as, for example, the MVN package tests [38, 39]) presents significant difficulties. To test for joint normality of the normalized sums of the mean of our PSI-process, we use the form of the covariance function and the fact that the normalized sums of the PSI-processes converge to the Ornstein–Uhlenbeck process. Here we rely on the representation of the Ornstein–Uhlenbeck process with discrete time as a first order autoregression with Gaussian white noise.

Let ε_0, ε_1, ε_2, ... be independent identically distributed standard normal random variables ($\varepsilon_j \in N(0,1)$, $j=\overline{0,+\infty}$). Let us define the Ornstein–Uhlenbeck process with discrete time and speed $\lambda > 0$ in recurrent form:

$$\begin{cases} u_0 = \varepsilon_0 \\ u_{n+1} = e^{-\lambda} u_n + \sqrt{1-e^{-2\lambda}}\,\varepsilon_{n+1}. \end{cases}$$

Obviously $u_j \in N(0,1), j=\overline{0,+\infty}$. You can write down

$$\begin{cases} \varepsilon_0 = u_0 \\ \varepsilon_{n+1} = \dfrac{u_{n+1} - e^{-\lambda} u_n}{\sqrt{1-e^{-2\lambda}}}. \end{cases} \quad (6)$$

We denote the number of elements in samples at each moment of discrete time as $N(j), j=\overline{0,+\infty}$ and set

$$u_j = \frac{1}{\sqrt{N(j)}} \sum_{i=1}^{N(j)} \hat{Y}_i(j), \quad j=\overline{0,+\infty}. \quad (7)$$

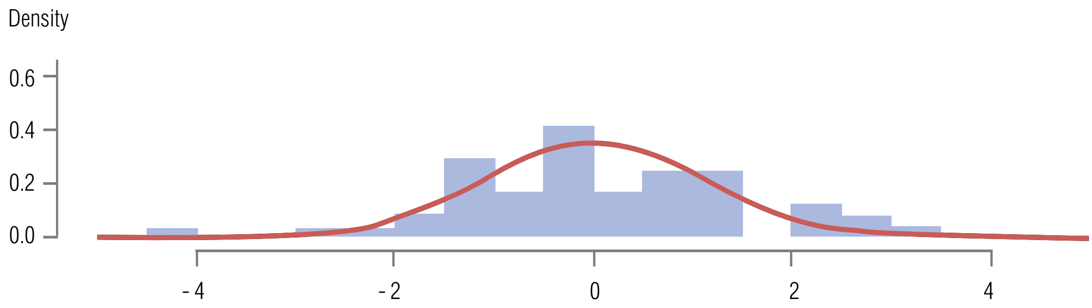

Fig. 10. Empirical distribution of values $\hat{\varepsilon}_j, j=\overline{0,47}$ and its approximation by the normal distribution density. Model distribution parameters $\boldsymbol{\mu}=0.0267$, $\sigma=1.1$.

Applying formulas (6) to quantities (7), we obtain estimates $\hat{\varepsilon}_0, \hat{\varepsilon}_1, \hat{\varepsilon}_2, \ldots$ for $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$. From the independence and normality of the values $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ the joint normality of values (7) follows.

The data consists of 48 samples (each of different lengths) corresponding to 48 discrete points in time. For each of them we set $\hat{Y}(j)$, $j=\overline{0,47}$, normalized sums of averages (7), estimates $\hat{\varepsilon}_0, \hat{\varepsilon}_1, \hat{\varepsilon}_2, \ldots$ (6). To do this, we need to identify the trend (shown above), estimate the parameter λ (we obtained it twice, $\lambda \approx 0.65$), the standard deviation was previously assumed to be constant $\sigma=0.22$. *Figure 10* shows the empirical distribution of the values $\hat{\varepsilon}_j, j=\overline{0,47}$ and its approximation by the normal distribution density. *Figure 11* shows a screenshot of the results of statistical tests for the normality of values $\hat{\varepsilon}_j$, $j=\overline{0,47}$ using the Kolmogorov–Smirnov, Shapiro, Andersen–Darling tests.

```
> ks.test(eps,"pnorm",mean=0.0267,sd=1.1)

        Exact one-sample Kolmogorov-Smirnov test

data:  eps
D = 0.096711, p-value = 0.7239
alternative hypothesis: two-sided

> shapiro.test(eps)

        Shapiro-Wilk normality test

data:  eps
W = 0.97975, p-value = 0.5684

> ad.test(eps)

        Anderson-Darling normality test

data:  eps
A = 0.25751, p-value = 0.7054
```

Fig. 11. Screenshot of the results of statistical tests of Kolmogorov–Smirnov, Shapiro, Andersen–Darling. The test results confirm the noise normality hypothesis (6).

Thus, the verifications carried out allow us to confirm the following hypotheses:

1) spacing (τ_i), $i=\overline{1,48}$ obey an exponential distribution law with parameter $\lambda \approx 0.65$;
2) each component of the discrete process $\hat{Y}(j)$, $j=\overline{0,47}$ is standard normal;
3) the observed copies of the random series $\ln(V(j))$ are independent (i is the copy number, j is the slice number);
4) the autocorrelation function can be approximated (*Fig. 8*) by an exponential function $e^{-\lambda \cdot s}$, with a parameter $\lambda \approx 0.65$, thus the correlation coefficient for a 6-step forecast (that is, six months ahead) can be set as $e^{-0.65 \cdot 6}=e^{-3.9}$;
5) the form of the partial autocorrelation function presented in *Fig. 9* indicates that $\hat{Y}(j)$ is conditionally uncorrelated with $\hat{Y}(k)$, $|k-j|>1$, $j=\overline{0,47}$, provided that the values of $\hat{Y}$ at moments of time strictly between k and i;
6) the Markov property of the discrete process $\hat{Y}(j)$, $j=\overline{0,47}$ follows from the joint normality and the

form of the covariance function. The Markov property is also confirmed by the form of the partial autocovariance function.

11. Forecast

The above checks allow us to establish the correspondence of the observed data to the PSI-process model. The Markov property of the observed discrete process $\hat{Y}(j), j=\overline{0,47}$ allows us to make a forecast only based on the latest distribution of the random variable V (the cost of 1 sq.m. of secondary residential real estate in the Admiralteysky district, in the 48th period corresponding to September 2015). The empirical distribution of the random variable $V(48)$ is satisfactorily approximated logarithmically normal distribution with parameters $\mu(0) = 4.64$, $\sigma = 0.22$ (p-value of the Kolmogorov–Smirnov test is 0.2534, see *Fig. 4*).

The mathematical expectation of the model distribution in the 48th period is equal to $e^{4.64+0.5\cdot 0.22^2} =$ $= 106.081$ thousand rubles for 1 sq. m. The median of the model distribution in the 48th period is equal to $e^{4.64} = 103.544$ thousand rubles for 1 sq. m. The mode of the model distribution in the 48th period is equal to $e^{4.64-0.22^2} = 98.652$ thousand rubles for 1 sq. m.

Note that the average price exceeds the most probable value by 7.5%, the minimum price is 1 sq. m. in the empirical sample in the 48th period 63.265 thousand rubles; maximum price 1 sq. m. in the empirical sample in the 48th period 172.556 thousand rubles.

The correlation function of the normalized sums of the discrete process $\hat{Y}(j), j=\overline{0,47}$ has an exponential form with the parameter $\lambda = 0.65$. Moreover, we obtained this value twice: as a parameter of the exponential approximating the correlation function, and as a parameter of the exponential distribution of spacing).

The correlation coefficient for the logarithm of a pair of arbitrary sections of the process $V(t)$, forming a two-dimensional random vector $(V(0), V(s))$, is equal to $\rho(\ln(V(0)), \ln(V(s))) = e^{-0.65\cdot s}$. Formulas (2)–(4) have the form:

$$Mode(V(s)|V(0) = v(0)) =$$
$$= \exp(0.00432 \cdot s + 4.443 + e^{-0.65\cdot s}(\ln(v(0)) - \mu(0) -$$
$$- 0.22^2(1 - e^{-2\cdot 0.65\cdot s})),$$

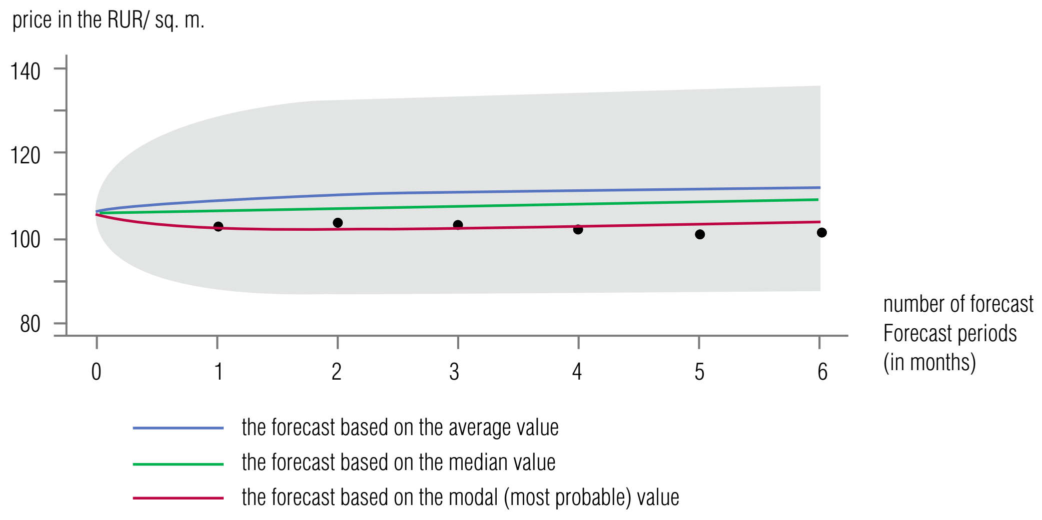

Fig. 12. Forecast for the price value $v(0) = 106.081$ thousand rubles per 1 sq. m. – the average price in the last observation period.
Gray background – corridor within one standard deviation.

$$Median(V(s)|V(0) = v(0)) =$$
$$= \exp(0.00432 \cdot s + 4.443 + e^{-0.65 \cdot s}(\ln(v(0)) - \mu(0))),$$

$$E(V(s)|V(0) = v(0)) =$$
$$= \exp(0.00432 \cdot s + 4.443 + e^{-0,65 \cdot s}(\ln(v(0)) - \mu(0)) +$$
$$+ \frac{1}{2}0.22^2(1 - e^{-2 \cdot 0.65 \cdot s})).$$

Formula (5), which describes the boundaries of the $V_{L,R}(s)$ error corridor (for example, within one standard deviation), takes the form:

$$V_{L,R}(s) = \exp\Big(0.00432 \cdot t + 4.443 +$$
$$+ e^{-0\ 65 \cdot s}\left(\ln(v(0)) - \mu(0)\right) \pm 0.22 \cdot \sqrt{1 - e^{-2 \cdot 0.65 \cdot s}}\Big).$$

For any property, price is 1 sq. m. which $V(0) = v(0)$ was set in the last period (as stated above, corresponds to September 2015), a forecast can be made for the next 6 periods.

Let us build a forecast for 6 periods (six months in advance) for a property that in the last period had a price of 106.081 thousand rubles for 1 sq. m. (average price for slice number 47).

Figure 12 shows the forecast for the price value $v(0) = 106.081$ thousand rubles per 1 sq. m.

In *Fig. 12*, dots mark the actual average prices for 1 sq. m. in the same set of real estate properties (mass market, Admiralteysky district), which are:

in October 2015 103.100 thousand rubles for 1 sq. m.,

in November 2015 103.492 thousand rubles for 1 sq. m.,

in December 2015 103.247 thousand rubles for 1 sq. m.,

in January 2016 102.027 thousand rubles for 1 sq. m.,

in February 2016 101.044 thousand rubles for 1 sq. m.,

in March 2016 102.046 thousand rubles for 1 sq. m.

Conclusion

1. The complex of checks carried out made it possible to establish that the studied data follows the PSI-process model.
2. *Figure 11* shows the forecast for a price equal to the average value in the last observation period. The black dots show the actually observed average price values of 1 sq. m. in the next 6 months, they are located near the modal forecast line, which serves as a good verification of the forecast based on the most probable value. The line of modal values indicates the most probable value of the future price, provided that the price in the last slice was equal to the average value. This is a consequence of the characteristic of the lognormal distribution of prices and the fact that the average values of the logarithms of prices follow the Ornstein–Uhlenbeck process.
3. The parameter $\lambda = 0.65$ indicates, among other things, the average time during which the price of a property in the observed market sector jumps; it is $1/\lambda = 1.54$ periods. In the example presented, the period is 1 month.
4. The main advantages of forecasting based on the characteristics of PSI-processes:

♦ the standard deviation of the forecast stabilizes over time at the level of constant variance of the random process, in contrast to moving average models that accumulate forecast error at each step;

♦ the ability to build a forecast not only for average values, but also for any object put up for sale in the last observation period at a certain price. ■

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